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12th Edition

Harold Cherry, FSA, MAAA

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TO OUR READERS:

Please check A.S.M.'s web site at **www.studymanuals.com** for errata and updates. If you have any comments or reports of errata, please e-mail us at **mail@studymanuals.com**.

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Introduction

To the student: Please read this Introduction. It contains important information.

The 12th edition of this manual has been revised and updated for the new syllabus in effect for the June 2017 and later exams. The major changes in the syllabus were:

- The topic of derivatives (except for interest rate swaps) was removed.
- A new study note on interest rate swaps was added.
- A new study note on duration and convexity was added.[1]
- A new study note on determinants of interest rates was added.

The 12th edition consists of 11 sections covering all of the material on the syllabus (Part I of the manual), followed by six original practice exams (Part II of the manual). Following each section or subsection in Part I, there are illustrative examples called "Stepping Stones" (see below). These examples are followed by a summary of the key concepts and formulas in the section, then by problems and solutions from actual past SOA/CAS examinations.

Goals of this Manual

- To explain the concepts of financial mathematics in a way that appeals to your intuition and common sense.
- To point out shortcuts and tricks that can get you to the answer more quickly.
- To warn you about common traps that students fall into and help you to avoid them.
- To provide you with hundreds of problems from past exams, with solutions.
- To provide you with original practice exams that will help prepare you for the real thing.

To highlight the concepts, tricks, shortcuts, and traps, you will see special symbols such as the following throughout the manual:

    ! **Concept Alert!**   **Shortcut Alert!**

    Trick Alert!   ● **Trap Alert!**

Problems

There is an old cliché in the real estate industry: What are the three most important factors in evaluating property? Answer: Location! Location! Location! If we had to say what the three most important factors are in studying for (and passing) this exam, they would be Problems! Problems! Problems! You must do a great variety of problems, preferably under time pressure, and especially as you get closer to the date of the examination.

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[1] The syllabus says that only Sections 1–4 of this study note are required reading. However, the author of this manual recommends that you read Section 5 as well.

Many of the problems in this manual are taken from past SOA and CAS examinations. We want to thank the two actuarial societies for their kind permission to publish these questions.

There are a number of points about past exam questions that you should be aware of:

- These questions, which date from the early 1980's, were created by different exam committees, under different syllabi, for exams of different length, etc. Thus, they can vary greatly in style, difficulty and emphasis of topics. In spite of this, you should find that solving prior exam problems, even very old ones, is helpful in preparing for the exam. It will expose you to a great variety of types of questions and approaches to solutions.

- This manual does not contain the questions and solutions available on www.BeAnActuary.org. Currently, this website includes the actual May 2005 and November 2005 exams (but note that these two exams were very easy) and well over 100 sample questions and solutions, many of which are from past exams. ***You should definitely get these additional practice problems.*** You can download them for free from the BeAnActuary website. We have not included them in this manual because we didn't want to waste space and expense by duplicating material that is freely available from another source.

- The actual exam consists of 35 questions in CBT (computer based testing) format, with a 3-hour time limit.

- The order of the past exam questions that follow each topic in this manual is from the most recent to the oldest.

- There is a code following each question in this manual from a past exam that tells you which exam the question came from.

- Some of the questions have answer choices in ranges, such as "(A) Less than 11.3% (B) At least 11.3% but less than 11.5%," etc. These questions almost always come from CAS exams, which used this style for many years. Virtually all questions on the SOA/CAS Exam FM/2 since 2000 have had specific answer choices, rather than ranges.

To make it easier to locate them, the sections of the manual with past exam questions and solutions have running heads that are shaded like this:

Past Exam Questions on Sections 1b to 1f

and

Solutions to Past Exam Questions on Section 1b to 1f

"Stepping Stones"

Following each section of the manual, you will find illustrative examples, with solutions, that are called "Stepping Stones." They are indicated by this heading:

Stepping Stones

These examples are original to the manual and have a level of difficulty that runs from easy to moderate. They have two purposes:

- To make sure you understand the material that you have just read.
- To serve as "stepping stones" to more difficult exam-level problems.

Original Practice Exams

In addition to the questions from past exams, the manual contains six original full-length practice exams, with solutions. These exams consist of 35 questions each, with a 3-hour time limit, just like the actual exam. The distribution of questions by topic follows the SOA syllabus.

Take these exams in the order given. We believe that Practice Exam 6 is the most difficult. As we receive feedback from students about the difficulty of the six exams, we will post the information on the ASM website at www.studymanuals.com.

Calculators

Begin using your calculator immediately. Become thoroughly familiar with its operation. It should become like a trusted friend to you after awhile.

Our advice is to get a financial calculator for the exam, such as the BA-35©, BA II Plus©, or BA II Plus Professional©. These calculators have special keys that are very useful for solving problem in financial mathematics. Which one should you use? Any of them will serve you well for the exam. However, Texas Instruments discontinued manufacturing the BA-35 in 2005. You really don't need the Professional model, unless you are willing to spend extra money for some additional features that aren't essential for the exam. For these reasons, we have based the Calculator Notes in this manual on the BA II Plus. You will find these notes at appropriate points in the manual, identified by this icon:

Please note that many students prefer to use one of the TI-30 MultiView series as their primary calculator because of the two-line display and superior algebraic capabilities. These students feel that they can even calculate certain interest functions, such as $a_{\overline{n}|}$, more quickly on the TI-30 series, even though the BA series has special keys for this function. They use the BA series as a backup calculator, mainly to calculate an unknown interest rate when the amounts of payments and their present value are given.

You can do a search on the Discussion Forum of the Actuarial Outpost (www.actuarialoutpost.com) to get other students' opinions on which calculators to use, and then make your own decision.

Studying for the Exam

Everyone is different. Some people like to study in the morning, some late at night. Some people can only study for a couple of hours before their brain begins to fry; others can study for long stretches. Some people can cram very effectively, but others need to cover the material over a period of several months. Almost everyone finds ways to procrastinate from time-to-time. You know your own study habits best, so build on your strengths and be aware of your weaknesses.

We recommend that you set up a study schedule right now. Determine how many hours you will realistically be able to put in between now and the exam. Many students use this rule-of-thumb: devote 100 hours of study time for each hour of exam time. For a 3-hour exam like FM/2, this would mean 300 hours. Of course, people vary considerably in how quickly they learn new material, so use this rule-of-thumb only as a rough guide.

When you set up your study schedule, be very aware of the SOA's allocation of questions by topic. The following table shows how many questions, on the average, you should expect for each topic, assuming that the SOA sticks to its own allocation when it constructs your particular computer-based exam. The table also shows which sections of the manual cover each topic.

Topic	Allocation by SOA	Midpoint of Allocation	Number of Questions (Based on a 35-Question Exam)	Sections of the Manual that Cover the Topic
A. Time Value of Money (Interest & discount rates, force of interest, equations of value, etc.)	10–15%	12.5%	4.375	Sections 1 & 2
B. Annuities (including arithmetic and geometric)	15–20%	17.5%	6.125	Sections 3 & 4
C. Loans (including interest and principal portion of payments, final payment questions, amortization, sinking funds, etc.)	15–20%	17.5%	6.125	Section 6
D. Bonds	15–20%	17.5%	6.125	Section 7
E. General Cash Flows and Portfolios (yield rates, dollar-weighted and time-weighted rates of returns, duration, convexity, spot rates, forward rates, yield curve, dividend discount model, etc.)	10–15%	12.5%	4.375	Sections 5, 8, 10a to 10g, 11a
F. Immunization	10–15%	12.5%	4.375	Sections 10h to 10k
G. Interest Rate Swaps	0–10%	5%	1.75	Section 11b
H. Determinants of Interest Rates	0–10%	5%	1.75	Section 9
Grand Total		100.0%	35	

Note especially that topics B, C, and D comprise 52.5% of the exam and that topics A, E, and F comprise 37.5% of the exam, for a total of 90%. This tells you where you should spend most of your time studying.

The above table is based on the syllabus for the June 2017 exam as posted on the BeAnActuary website in mid-January 2017, when this manual was published. Please note that the final syllabus will be posted on that website at the end of January 2017 or in early February. You should check the final syllabus to see whether there have been any changes. Also, if you are taking the exam in August 2017 or later, check the syllabus for the particular month and year of your exam, since the exam committee does make changes from time to time.

As you follow your schedule, will you fall behind from time-to-time? Of course. But if you have a schedule, at least you will know how far behind you are. This should spur you on to catch up.

Should you do all of the problems in a sub-section before you move on to the next sub-section? Our advice is to do as many problems as you can, but to move on if you are falling behind schedule; hopefully, you will catch up later.

You should leave room at the end of your schedule for at least three or four weeks of doing nothing but solving problems (and maybe a little bit of review of the topics you are having difficulty with). Be strict with yourself and work "by the clock." It's not important that you get the correct answer to a problem the first time that you do it. What *is* important is that you try the problem again a couple of days later and get it right the second (or even the third) time. (Mark off the problems that you don't get right and do batches of these problems a few days later.) This shows that you have really mastered the points being tested.

"Hitting a Brick Wall": Sections 3 and 4

Some students find that when they attempt to cover the topics in Section 3 and 4, it's like "hitting a brick wall." There is a lot of material in these two sections and it can be daunting if you are encountering it for the first time. If this happens to you, we have a couple of suggestions:

- Don't knock yourself out if you don't understand something. Move on to the next section and come back later on. You will find that as you spend more time on the material, you will mature in your understanding. Concepts that seemed difficult at first will seem easier when you revisit them.
- Don't try to do all of the problems at the end of each section the first time around. Do the first few, since they are the most recent, or do only the odd or even problems. Come back later and do the rest of the problems (or most of them). They will seem easier the second time around.

It's true that there are a lot of problems in these two sections, about 200 in all, but this shows how important these topics have been on past exams.

Taking the Exam

The single most important rule about taking the exam is to *keep moving*. Don't get bogged down on any one question. Try to look at every question at least once. Be aware of your progress throughout the exam. Try to spend your time on the problems that are easier for you and drop the ones that are giving you too much trouble.

Since this exam has 35 questions and a 3-hour time limit, you have an average of about 5 minutes to spend on each question. You may be tempted to continue spending time on a problem because you have already invested, say, 5 or 6 minutes in it. We strongly suggest that you develop the discipline to drop the question at this point and move on. There may be an easier question waiting for you later on—and getting it right counts just as much as getting a hard question right. And you may have time to go back to the question later on and knock it off.

Your Comments

We welcome your comments, criticism, suggestions, and reports of any errata that you may find. Please e-mail us at *mail@studymanuals.com*

Acknowledgments

My thanks to Wafaa Shaban, Lecturer and Director of the Actuarial Program at the University of North Carolina at Charlotte, who meticulously reviewed all of the new material in the 12th edition, noted many corrections and made valuable suggestions for improvements. Of course, any remaining errors and defects are entirely my own responsibility. Thank you, Wafaa.

My thanks to Rick Gorvett, currently Staff Actuary at the Casualty Actuarial Society, who was my co-author for previous editions of this manual. In addition to making significant contributions to these editions, Rick was the point man for fielding questions from students about the manual, a task he handled with agility, clarity, and good humor. Thank you, Rick.

My thanks to Paul Anagnostoupoulos of Windfall Software, who, as usual, rapidly keyboarded my messy scribblings into intelligible English and precise mathematical notation. Thank you, Paul.

Finally, my thanks go to my long-suffering wife, Maida, who had to compete with this manual for my time, usually unsuccessfully, over a period of several months. I'm sorry, Maida, that I had to take time out from my supposed "retirement" to once again work on the manual. I will convey your exhortations to the SOA that they not change the syllabus again for at least 25 years, but based on their track record, I think it's a vain hope. Thank you so much for your patience, Maida.

———————————

I wish you good studying and good luck on the exam. May the force of interest be with you.

Harold Cherry

PART I

Financial Mathematics

1

Interest Rates and Discount Rates

§ 1a. Basic Concepts

§ 1a(i) Accumulation Function and Effective Rate of Interest

Imagine a fund growing at interest. It would be very convenient to have a function representing the amount in the fund at any time t. The function $a(t)$ is defined as the accumulated value (AV) of the fund at time t of an initial investment of \$1.00 at time 0. $a(t)$ is called the "accumulation function."

Consider the following accumulation functions. Can you think of any real-life situations where you might encounter them?
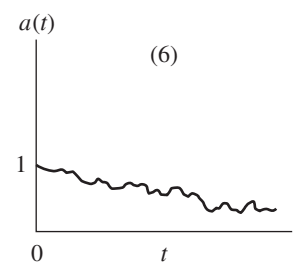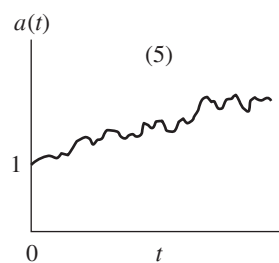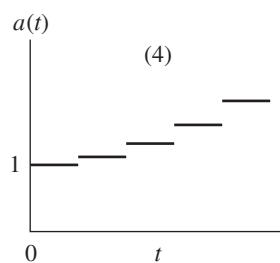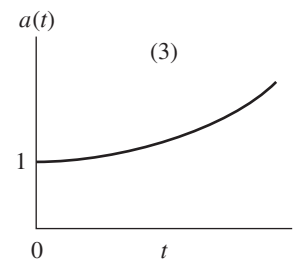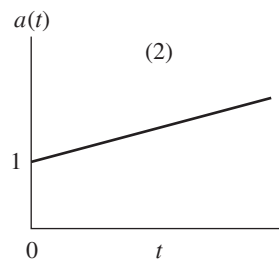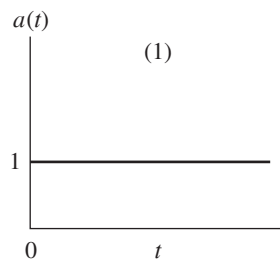
(1) This is the accumulation function for money put in a piggy bank or under the mattress. It might also represent "a friendly loan from my father-in-law" or a checking account where you get no interest.

(2) This is so-called "simple interest," where the accumulation function is linear. But we will see that "simple interest" isn't so simple at all.

(3) This accumulation function is an exponential. As we shall see, this is referred to as "compound interest," where the fund earns interest on the interest at a constant rate.

(4) This is the accumulation function for an account where you are credited with interest only at the end of each interest period.

(5) This is an accumulation function for a fund that grows at a varying rate. An investment in stocks is often given as an example.

(6) This is the stock you bought last month.

What are the properties of an accumulation function? By definition, $a(0) \equiv 1$. Other than that, anything goes. But the accumulation functions that we will generally deal with in this course will also have the properties of being (1) continuous and (2) increasing.

Suppose we want to measure the rate of growth of a fund in, say, the tth year. Let's say that the accumulation function looks like the following graph in the tth year. (Remember that the tth year runs from time $(t-1)$ to time t, just as the first year runs from time 0 to time 1.)
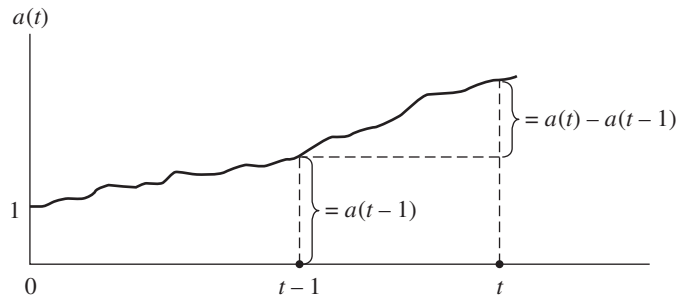
The *amount* of growth in the tth year (i.e., the interest earned) is $a(t) - a(t-1)$. The *rate* of growth (based on the amount in the fund at the beginning of the year) is:

$$\frac{a(t) - a(t-1)}{a(t-1)}$$

(Note that the amount in the fund at the beginning of the tth year is the same as the amount in the fund at the end of the $(t-1)^{st}$ year, namely, $a(t-1)$, as long as no new investments are made.) This rate of growth is called the "effective rate of interest" and has the symbol i_t. So we have:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

We will also define an "amount function," $A(t)$, as the AV at time t of k invested at time 0 (rather than 1 invested at time 0). Obviously, $A(t) = ka(t)$. Mathematically, we need only one of these two functions, but having both can be handy. Given the definition of $A(t)$, the effective rate of interest can also be defined as:

$$i_t = \frac{A(t) - A(t-1)}{A(t-1)}$$

because substituting $A(t) = ka(t)$ and $A(t-1) = ka(t-1)$, the k's cancel and we're right back to the original definition.

Using this version of the definition of i_t, go ahead and solve for $A(t)$. You should get the following:

$$A(t) = A(t-1) + i_t A(t-1)$$

or $A(t) = (1 + i_t)A(t-1)$

How would you explain this?

The answer is as follows: The fund at the end of the tth year is equal to the fund at the beginning of the year plus the interest earned during the year.

Note that the interest earned during the year is the effective rate of interest multiplied by the fund at the beginning of the year. This is consistent with the definition of the effective rate of interest.

Stepping Stones

EXAMPLE 1

You are given that the amount function is $A(t) = 4t^2 + 8t + 4$. What is the accumulation function?

SOLUTION

First, determine k: $A(0) = k = 4$. Then $A(t) = ka(t) = 4a(t)$ and $a(t) = (1/4)A(t) = (1/4)(4t^2 + 8t + 4) = t^2 + 2t + 1$.

EXAMPLE 2

For the amount function in Example 1, what is the total interest earned in the 2nd, 3rd, and 4th years?

SOLUTION

Total interest $= A(4) - A(1) = 100 - 16 = 84$.

EXAMPLE 3

For the amount function in Example 1, what is the effective rate of interest in the 5th year?

SOLUTION

$$i_5 = [A(5) - A(4)]/A(4) = (144 - 100)/100 = 44\%$$

EXAMPLE 4

John invests X in a fund growing in accordance with the accumulation function implied by the amount function in Example 1. Edna invests X in another fund growing in accordance with the accumulation function implied by the amount function $A(t) = 4t^2 + 2$. When does Edna's investment exceed John's?

SOLUTION

We want to compare the two funds assuming that equal investments of X are made at time 0. From Example 1, John's accumulation function is $t^2 + 2t + 1$. For Edna, $k = 2$ and the accumulation function is $2t^2 + 1$. To determine when Edna's investment exceeds John's, we set:

$$X(2t^2 + 1) > X(t^2 + 2t + 1)$$

which reduces to:

$$t^2 - 2t > 0 \quad \text{or} \quad t(t - 2) > 0$$

Thus, Edna's fund exceeds John's after 2 years.

§ 1a(ii) Simple Interest

Let's consider so-called "simple interest." What are the effective rates of interest for years 1, 2, 3, ..., t? Under simple interest, the accumulation function is linear. We will use i as the linear constant:

$$a(t) = 1 + it$$

We have the following by definition of i_t:

$$i_1 = \frac{a(1) - a(0)}{a(0)} = \frac{(1+i) - 1}{1} = i$$

$$i_2 = \frac{a(2) - a(1)}{a(1)} = \frac{(1+2i) - (1+i)}{1+i} = \frac{i}{1+i}$$

$$i_3 = \frac{a(3) - a(2)}{a(2)} = \frac{(1+3i) - (1+2i)}{1+2i} = \frac{i}{1+2i}$$

$$\vdots$$

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} = \frac{(1+it) - [1+i(t-1)]}{1+i(t-1)} = \frac{i}{1+i(t-1)}$$

For each year, the numerator (the interest earned) is a constant i, since $a(t)$ is linear. The denominator (the amount in the fund at the beginning of the year) is increasing, which means that the effective rate of interest decreases with time.

So we see that "simple interest" isn't simple at all. A linear accumulation function implies a decreasing effective rate of interest. (In fact, it decreases hyperbolically!)

In financial transactions, simple interest is often used for fractions of an interest period for convenience. (It is a bit confusing to use i as the linear constant, since it is the effective rate of interest for the first year only. It would be better to use another constant, as in $a(t) = 1 + kt$, but the general practice is to use i.)

Stepping Stones

EXAMPLE 1

Money doubles in 10 years at a simple interest rate of i per annum. Determine i.

SOLUTION

We are given that 1 accumulates to 2 in 10 years at a simple interest rate i:

$$1 + 10i = 2$$
$$i = .10 = 10\%$$

EXAMPLE 2

John deposits $100 at a simple interest rate of i per annum. At the end of 10 years, he has $150 in his account. Lolita deposits $200 at a simple interest rate of $2i$ per annum. How much is in her account at the end of 20 years?

SOLUTION

First, determine i:

$$100(1 + 10i) = 150$$
$$i = .05 \quad \text{and} \quad 2i = .10$$

At the end of 20 years, Lolita has $200[1 + (20)(.10)] = \$600$ in her account.

§ 1a(iii)   The Accumulation Function in Terms of i_t

Suppose we are given that the effective rate of interest is 5% in the first year and 6% in the second year. We invest $1.00 at time 0. How much is in the fund at the end of one year? Obviously, $1.05. How much is in the fund at the end of two years? We start with $1.05 at the beginning of the second year and we earn interest on it at 6%, so:

$$a(2) = 1.05 + .06(1.05)$$

or $a(2) = (1.05)(1.06)$

In symbols:

$$a(2) = (1 + i_1)(1 + i_2)$$

If we continue this process for t years, we have:

$$a(t) = (1 + i_1)(1 + i_2) \ldots (1 + i_t)$$

or using the symbol $\prod$ for product:

$$a(t) = \prod_{j=1}^{t}(1 + i_j)$$

Note that each factor in this chain multiplication has the effect of (a) bringing the fund at the beginning of the year to the end of the year without interest (this is the "1" in $(1 + i_j)$) and (b) adding interest on this amount at the effective rate (this is the "i_j" in $(1 + i_j)$).

Stepping Stones

EXAMPLE 1

Elsie deposits $500 in a bank account that earns interest at a 5% effective rate in year 1. At the end of year 2, Elsie has $552.30 in her account. What effective rate of interest did she earn in year 2?

SOLUTION

$$(500)(1.05)(1 + i_2) = 552.30$$
$$i_2 = 5.20\%$$

EXAMPLE 2

The effective rate of interest in year t is given as $i_t = k + .01(t - 1)$, where k is a constant. You are given that the accumulated value of $100 at the end of 2 years is $112.35. Determine the effective rate of interest for year 2.

SOLUTION

$$100(1 + i_1)(1 + i_2) = 112.35$$
$$(1 + k)(1 + k + .01) - 1.1235 = 0$$
$$k^2 + 2.01k - .1135 = 0$$

Using the quadratic formula, you should get the positive root of k as $k = .0550$ (to 4 decimals). Then $i_2 = k + .01 = .0650 = 6.50\%$.

§1a(iv)   Compound Interest

Let's take the case where i_j is a constant i. The chain product collapses into:

$$a(t) = (1 + i)^t$$

This special case is called "compound interest." (Any chain multiplication, even where the i_j's are not constant, involves earning interest on the previous interest, or "compounding." But you will often see the term "compound interest" used for the case of a constant effective rate.) Most of the problems for this exam are based on the assumption of compound interest.

Stepping Stones

EXAMPLE 1

Mary deposits money in a bank that compounds interest at i effective per annum. At the same time, Sam deposits the same amount as Mary in another bank that compounds interest at $(i + .01)$ effective per annum. At the end of 5 years, Sam has as much in his account as Mary will have at the end of 10 years. Determine i.

SOLUTION

$$(1 + i + .01)^5 = (1 + i)^{10}$$

Taking the 5$^{\text{th}}$ root of each side, we get:

$$(1.01 + i) = (1 + i)^2$$
$$i^2 + i - .01 = 0$$

Solving the quadratic for the positive root of i, you should get $i = .0099$ to 4 decimals, or $i = .99\%$.

EXAMPLE 2

Chauncey makes an investment of $1,000 that will earn interest at an effective annual rate of 5% for 5 years. Abigail makes an investment of $1,000 that will earn effective annual rates of 3%, 4%, 5%, 6%, and 7% in years 1, 2, 3, 4, and 5, respectively. Abigail boasts to Chauncey, "My investment is better than yours. I will have more than you 5 years from now." Is Abigail right?

SOLUTION

At the end of 5 years, Chauncey will have $(1,000)(1.05)^5 = \$1,276.28$ and Abigail will have $(1,000)(1.03)(1.04)(1.05)(1.06)(1.07) = \$1,275.70$. Abigail is wrong: She will have slightly less ($.58) than Chauncey.

§ 1a(v) Present Value

So far, we've been talking about the AV of a fund, i.e., how much is in the fund after t years, if we invest a given amount today. Consider the "opposite" question: How much should we invest today in order to have a given amount, say $1.00, at the end of t years? The amount that we should invest is called the **present value** (PV) of $1.00 due in t years.

Why would we be interested in present values? Isn't the real-life question, "How much is in a fund at a time t if we invest a given amount today?"

Suppose that in your personal financial life, you need to save or invest money that will grow to a specified amount in a specified number of years. For example, you may want to start saving now to buy a car for $20,000 in two years, or to make a down payment of $25,000 on a home in 3 years, or to pay a year's college expenses of $30,000 in 8 years. All of these questions involve finding the PV, i.e., the amount invested today that will grow to the desired amount in the desired time.

Also, in your work as an actuary, you will often have to determine how much someone should pay now in order to receive future benefits, such as death benefits under a life insurance policy, or lifetime income under an annuity contract or pension plan. (The payments, or "investments," and the benefits, or "accumulated values," may be paid in installments in many cases. Also, these examples involve contingencies, i.e., probabilities of future events. For this exam, we are only concerned with payments that are certain to be made.) So you can see that the concept of present value is *very important* in actuarial work.

Consider the general accumulation function $a(t)$. We want to determine how much to invest today in order to have $1.00 in t years. We will designate this amount as (PV). Since we require that (PV) grows to $1.00 in t years, we have:

$$(PV)a(t) = 1$$

$$(PV) = \frac{1}{a(t)}$$

The function $\frac{1}{a(t)}$ is called the "discount function." As we will see in the next section, "discounting" a future amount means the same thing as finding its present value. In particular, if $a(t) = (1+i)^t$ (the compound interest case), we have:

$$(PV) = \frac{1}{a(t)} = \frac{1}{(1+i)^t} = (1+i)^{-t}$$

Because the term $(1+i)^{-1}$ comes up so often (i.e., the PV of $1.00 due in a year), a special symbol has been invented for it, namely, v. So we have:

$$v = \frac{1}{1+i} = (1+i)^{-1}$$

(PV) of $1.00 due in t years$= v^t = (1+i)^{-t}$.

Stepping Stones

EXAMPLE 1

What deposit made today will provide for a payment of $1,000 in 1 year and $2,000 in 3 years, if the effective rate of interest is 7.5%?

SOLUTION

$$PV = 1{,}000v + 2{,}000v^3 \text{ at } 7.5\% = \$2{,}540.15$$

$$(v = 1.075^{-1})$$

EXAMPLE 2

A fund credits interest at a constant effective rate. The present value of $1.00 due in n years is $.50. Three deposits of $1.00 each are made at times 0, n, and 2n. What is the accumulated value of these deposits at time 3n?

SOLUTION

Since $v^n = 0.5$, $(1+i)^n = 2$. The AV of the 3 deposits is $(1+i)^{3n} + (1+i)^{2n} + (1+i)^n = 2^3 + 2^2 + 2 = 14$.

EXAMPLE 3

The accumulation function is given as $a(t) = .1t^2 + .2t + 1$. Determine the present value of $5 paid at time 5 and $10 paid at time 7.

SOLUTION

$$PV = 5/a(5) + 10/a(7) = 5/4.5 + 10/7.3 = \$2.48$$

Calculator Notes

It is essential that you use a financial calculator to get numerical results. As we noted in the Introduction, Calculator Notes for the BA II Plus will appear at appropriate points in this manual. If you are using another calculator, do the problems in these notes anyway, and check that you got the same answers.

Calculator Notes #1 begins on the following page.

Calculator Notes #1: Formatting; Present Values and Future Values

"Calculator Notes" will appear at appropriate points in this manual. These notes will cover the functions of the Texas Instruments BA II Plus calculator that are essential for Exam FM/2. (As noted in the Introduction, the BA-35 was discontinued by TI in 2005, although it is still an authorized calculator for the actuarial exams.)

These notes will not cover functions that you don't need for the exam (for example, the trig, statistical or depreciation functions).

In addition to the Calculator Notes in this manual, you should print out the review of calculator functions for the BA II Plus written by Sam Broverman and available on the BeAnActuary.org website. Also, you should have the TI guidebook as a general reference. If you don't have it, you can download it from the TI website.

Primary and Secondary Functions

The calculator has primary and secondary functions. The primary functions are printed on the keys themselves, for example, ☒ for "multiply." The secondary functions are shown in yellow *above* the keys. For example, the secondary function of the ☒ key is x!.

The secondary functions are accessed by pressing the ☐2nd☐ key (which is the only yellow key) and then the desired key. For example, to compute 5!, use the following sequence of keystrokes: 5 ☐2nd☐ [x!]. You should get 120.

Note that we have indicated the primary function ☒ by using a box and the secondary function [x!] by using brackets.

Formatting the Calculator

The calculator was pre-set to certain defaults when it left the factory, but these settings may have been changed since then. Let's format the calculator so that we can all start from the same place.

To access the Format mode, press ☐2nd☐ [FORMAT]. (The [FORMAT] key is the middle key in the bottom row.)

You can scroll the various formats by pressing the ☐↓☐ or ☐↑☐ keys in the top row. Scrolling down, note that the formats are "DEC," "DEG," "US" (for dates), "US" (for number separation), "Chn," and back to "DEC." We are only interested in a couple of these formats for Exam FM/2.

Scroll to "DEC." This controls the number of decimal places shown in the display. The factory default is two decimal places but you can set the calculator to show from 0 to 9 places. (The calculator uses 13 places internally, regardless of what is shown in the display.)

Let's say we want the display to show 4 decimal places. While "DEC" is in the display, press 4 ☐ENTER☐. Let's check that this has worked. Leave the [FORMAT] mode by pressing ☐2nd☐ [QUIT]. (The [QUIT] key is in the upper left-hand corner.) This puts us in the standard calculator mode. Now multiply two numbers, such as 2.6835 ☒ 5.625 ☐=☐. The answer shows to 4 decimals (rounded) as 15.0947.

Leave your calculator set to display 4 decimals. This is enough accuracy for most questions on the exam. (Remember that internally, 13 places are carried.) But you can set it for fewer or more places if you wish. It's a matter of personal preference: After you've used the calculator for awhile, you can decide for yourself.

Let's return to the Format mode by pressing ☐2nd☐ [FORMAT] again. Now scroll down to "DEG." This controls whether angles are expressed in degrees or radians, which doesn't concern us for this

exam. So we scroll down to "US 12-31-1990." This controls how dates are shown (i.e., whether month or day comes first). We're not going to be using dates very much, so scroll down to "US 1,000.0000." This controls how numbers are separated. For example, in US format, "one thousand" with 4 decimals is shown as 1,000.0000. In European format, commas become decimal points and decimal points become commas, so "one thousand" is shown as 1.000,0000. You should set the calculator in US format. In case it's in European format (shown as "EUR"), you can change it as follows: 2nd [SET]. (The [SET] key is secondary to the ENTER key.) If you keep pressing 2nd [SET], the display will alternate between "US" and "EUR." Leave it at "US."

Finally, scroll down to "Chn." This controls the calculation method, i.e., either "Chain" or "AOS" (Algebraic Operating System). To alternate between "Chn" and "AOS," press 2nd [SET]. Leave the setting at "Chn."

What's the difference between the Chain and AOS calculation methods? The Chain method is probably what you are most familiar with in a calculator. For example, if you press 2 ⊞ 5 ⊠ 3 ⊜, you expect the answer to be 21, i.e., the calculator first adds 5 to 2 and then multiples the result by 3. The AOS method follows the hierarchy of mathematical operations: multiplication and division are completed before addition and subtraction. Thus, for the same keystrokes in the AOS mode, the answer will be 17: The calculator multiples 5 by 3 first and then adds 2. You should leave the calculator in "Chn" mode.

You now have completed formatting the calculator. Press 2nd [QUIT] to return to standard calculator mode.

Accumulated Values and Present Values

1. What is the accumulated value of $1,537 at the end of 8 years at an effective annual interest rate of 7%?

There are two ways of computing this:

a. Using the regular keys

The exponential key y^x is located in the 4th column, 5 keys down. You'll be using it a lot, so get to know where it is.

The answer to this question is $1,537(1.07)^8$. Use these keystrokes:

1.07 y^x 8 ⊠ 1,537 ⊜

The display should show 2,640.8522.

Note that it is not necessary to press ⊜ after entering 1.07 y^x 8. When you press ⊠, the exponentiation is automatically done and the result is ready to be multiplied by the next value you enter. (Some guides include this equal sign in the keystrokes. This doesn't affect the final result, but it's not necessary.)

b. Using the TVM keys

"TVM" stands for "time value of money." This simply means that $1.00 in your greedy little hand today is worth more than $1.00 paid to you at a future time. (See the first page of Chapter 2 of this manual.)

The TVM keys are the keys in the 3rd row: N, I/Y, PV, PMT and FV. For simple accumulation and present value problems, it really isn't necessary to use these keys: It's easy and direct to use the y^x key, as we did above. However, the TVM keys are very important when it comes to handling annuities, loans, mortgages, bonds, etc. (covered in Chapter 3 and later), so we may as well get familiar with them now.

N is the number of years (or periods).

I/Y is the interest rate per year (or per period).

PV is the present value.

PMT is the amount of each payment of an annuity. (We won't use this key until Chapter 3.)

FV is the future value. (This is the same as what we have been calling the accumulated value, or AV.)

There are two very important "hidden" settings that affect the TVM calculations. Press 2nd [P/Y] ([P/Y] is secondary to the I/Y key) and the display will show "P/Y = ." "P/Y" stands for the number of payments per year. Now press the arrow key ↓. "C/Y = " will show in the display. "C/Y" stands for the number of times that interest is compounded per year.

We strongly recommend that you set P/Y equal to 1 and always keep it at 1.[1]

This is because it's very easy to forget what value is in the P/Y register. If you know that it's always set to 1, you are much less likely to make calculation errors. (This will become clearer when we get to Section 4.)

This is how to set P/Y equal to 1: Press 2nd [P/Y], then press 1 ENTER. C/Y will also automatically be set to 1. (You can check this by pressing ↓.)

To get back to the standard calculation mode, press 2nd [QUIT].

Trap Alert!

When you take the actual exam, the proctor will probably reset the calculator to its factory defaults by pressing 2nd [RESET]. (The [RESET] key is secondary to the +/− key.) If your calculator is reset, make sure that [P/Y] = [C/Y] = 1, because the default for some older models of the BA-II Plus is 12. Also, resetting the calculator will set two decimal places to show in the display. If you want to display a different number of places, you will have to set that as well.

Now we're ready to do problem 1. We have to enter the values of N, I/Y and PV, and then compute FV. But first make sure that you clear any current values in the TVM registers by pressing 2nd [CLR TVM]. ([CLR TVM] is secondary to the FV key. Pressing it will *not* change the settings of P/Y and C/Y.) Then use these keystrokes:

8 N 7 I/Y 1,537 +/− PV CPT FV

The answer should appear as 2,640.8522, which agrees with the previous answer. (Pressing the +/− key, which is in the bottom row, changes the sign of the value in the display. We will explain why we enter −1,537 as the PV in the comments below.)

Some important comments:

- The data can be entered in any order in the TVM keys. For example, we could have entered the interest rate first, then the number of years and then the PV. If we then press CPT FV, we would get the same answer as above.
- The effective rate of interest is entered as a percent in I/Y, so 7% is entered as 7, not .07. Similarly, a rate like 3.4562% would be entered as 3.4562.
- After we enter the data, we compute the unknown by pressing CPT and then the value that we want, in this case FV.
- Here is an explanation of why we enter the PV as a negative amount: We will take the point of view of a depositor, a lender, etc., i.e., someone who has a cash flow *out* in the amount of 1,537 at time 0. At the end of 8 years, we expect to receive a cash flow *in*, as a withdrawal from the bank, the repayment of the loan by the borrower, etc. The calculator expects us to

---

[1] C/Y will automatically be set to whatever P/Y is set to. It can also be set independently of P/Y.

enter a cash flow *out* as a negative value. When we press $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$, the calculator computes the cash flow *in* that we receive at the end of 8 years as a positive value.

Of course, we could reverse roles. If we take the point of view of the bank, the borrower, etc., 1,537 is the amount we *receive* at time 0, i.e., it is a cash flow *in,* and we would enter it as a positive value. When we press $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$, the calculator computes FV as a negative amount, since it is a cash flow *out*, i.e., the amount the bank, the borrower, etc., repays the investor.

It really doesn't matter which point of view we take, as long as we are consistent within the same problem. (In Calculator Notes, we will sometimes take one point of view and sometimes the other. If you happen to take the other point of view, our answers will have opposite signs.)

- Sometimes you may forget what values are in the TVM registers. You can check a register by pressing $\boxed{\text{RCL}}$ and then the register you want. Try this now.
- If there is an incorrect value in a register, simply enter the correct value in the display and then press the register key.
- To check the value currently in the P/Y register, press $\boxed{\text{RCL}}$ $\boxed{\text{2nd}}$ [P/Y]. For the value of C/Y, press $\boxed{\downarrow}$.
- It's essential to clear the TVM registers before you start a new problem. You do this by pressing $\boxed{\text{2nd}}$ [CLR TVM]. *Even if you turn the calculator off, the "garbage" that you left in TVM will still be there when you turn the calculator back on! Try it and see! So you must clear these registers for each new problem.*

Now that you have done this problem using the TVM keys, and have read all of the comments, we wouldn't blame you for asking, "Why go through all of this when we could simply use the $\boxed{y^x}$ key?" The answer is that, as we noted before, you will be using TVM *big time* starting with Chapter 3, so you may as well get used to it now. With a little practice, using TVM will become second nature to you. Also, some problems, like finding an unknown interest rate or an unknown time are easier to do using the TVM keys.

Now let's continue using the calculator to solve problems.

2. What is the PV of $1,250 due in 8 years at an effective annual interest rate of 5.23%?

a. Using the regular keys

1.0523 $\boxed{y^x}$ 8 $\boxed{+/-}$ $\boxed{\times}$ 1,250 $\boxed{=}$

The answer is $831.37 to the nearest $.01.

b. Using the TVM keys

Make sure that you have kept P/Y and C/Y equal to 1, as we recommended.

$\boxed{\text{2nd}}$ [CLR TVM] (to clear the TVM registers)

8 $\boxed{\text{N}}$ 5.23 $\boxed{\text{I/Y}}$ 1,250 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$

The answer is -831.37. If you had entered 1,250 as a negative, the answer would show as a positive value.

Unknown interest rate

3. $10,250 accumulates to $23,237 in 13 years. What is the effective annual interest rate?

a. Using the regular keys

The equation of value for this problem is:

$$10,250(1+i)^{13} = 23,237$$

Solving for i:

$$i = \left(\frac{23{,}237}{10{,}250}\right)^{1/13} - 1$$

Note: The calculator has 10 storage registers numbered 0 to 9. To store a value that appears in the display, let's say in register 0, simply press $\boxed{\text{STO}}$ 0. The value will be stored in register 0 and will also continue to be shown in the display. If you want to recall this value later, press $\boxed{\text{RCL}}$ 0 and it will be shown in the display. It will also remain in the register.

We can use the following keystrokes:

13 $\boxed{1/x}$ $\boxed{\text{STO}}$ 0 23,237 $\boxed{\div}$ 10,250 $\boxed{y^x}$ $\boxed{\text{RCL}}$ 0 $\boxed{-}$ 1 $\boxed{=}$

The answer is .0650 or 6.50%.

Note that we computed 1/13 as the reciprocal of 13, stored it, and later recalled it to use as the exponent in the $\boxed{y^x}$ operation.

b. Using the TVM keys

$\boxed{\text{2nd}}$ [CLR TVM] *(From now on, we will not show this step. We'll assume that you do it at the beginning of each new problem.)*

13 $\boxed{\text{N}}$ 10,250 $\boxed{+/-}$ $\boxed{\text{PV}}$ 23,237 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$

The answer is 6.50% to 2 decimals, as before.

Unknown time

4. At an effective rate of interest of 8% per annum, the present value of $100,000 due in X years is $65,322. Determine X.

a. Using the regular keys

The equation of value is $65{,}322(1.08)^X = 100{,}000$. Solving for X:

$$1.08^X = \frac{100{,}000}{65{,}322}$$

$$X = \frac{\ln\left(\frac{100{,}000}{65{,}322}\right)}{\ln 1.08}$$

The natural log key $\boxed{\text{LN}}$ is located in the first column of keys. We can use the following keystrokes:

1.08 $\boxed{\text{LN}}$ $\boxed{\text{STO}}$ 0 100,000 $\boxed{\div}$ 65,322 $\boxed{=}$ $\boxed{\text{LN}}$ $\boxed{\div}$ $\boxed{\text{RCL}}$ 0 $\boxed{=}$

The answer is 5.5332 years.

b. Using the TVM keys

8 $\boxed{\text{I/Y}}$ 65,322 $\boxed{+/-}$ $\boxed{\text{PV}}$ 100,000 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{N}}$

The answer is $N = 5.5332$ years.

5. How long does it take money to double at 5% effective?

a. Using the regular keys

$$1.05^x = 2$$

$$x = \frac{\ln 2}{\ln 1.05}$$

1.05 $\boxed{\text{LN}}$ $\boxed{\text{STO}}$ 0 2 $\boxed{\text{LN}}$ $\boxed{\div}$ $\boxed{\text{RCL}}$ 0 $\boxed{=}$

The answer is 14.2067 years.

b. Using the TVM keys

5 I/Y 1 +/− PV 2 FV CPT N

The answer is N = 14.2067 years.

Summary of Concepts and Formulas in Sections 1a(i) to 1a(v)

(1) The accumulation function $a(t)$ is the AV at time t of \$1.00 invested at time 0.

(2) The effective rate of interest in the tth year is based on the amount in the fund at the beginning of the year:

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)}$$

(3) Under "simple interest":

$$a(t) = 1 + it$$

$$i_t = \frac{i}{1 + i(t-1)}$$

(4) In terms of effective rates of interest:

$$a(t) = (1 + i_1)(1 + i_2) \cdots (1 + i_t)$$

(5) Under "compound interest," the effective rate of interest is a constant in all years:

$$i_t = i \quad \text{and} \quad a(t) = (1 + i)^t$$

(6) (a) For any accumulation function $a(t)$, the present value of \$1.00 due in t years is:

$$PV = \frac{1}{a(t)}$$

(b) Under compound interest:

$$PV = \frac{1}{(1+i)^t} = (1+i)^{-t} = v^t$$

Past Exam Questions on Sections 1a(i) to 1a(v)

1. Sally has two IRA's. IRA #1 earns interest at 8% effective annually and IRA #2 earns interest at 10% effective annually. She has not made any contributions since January 1, 1985, when the amount in IRA #1 was twice the amount in IRA #2. The sum of the two accounts on January 1, 1993 was $75,000. Determine how much was in IRA #2 on January 1, 1985. [CAS 5/93 #6]

   (A) <$12,750   (B) $12,750/$13,000   (C) $13,000/$13,250   (D) $13,250/$13,500   (E) ≥ $13,500

2. Money accumulates in a fund at an effective annual interest rate of i during the first 5 years, and at an effective annual interest rate of $2i$ thereafter.

   A deposit of 1 is made into the fund at time 0. It accumulates to 3.09 at the end of 10 years and to 13.62 at the end of 20 years.

   What is the value of the deposit at the end of 7 years? [SOA 5/93 #2]

   (A) 1.90   (B) 1.98   (C) 2.06   (D) 2.14   (E) 2.23

3. At an effective annual interest rate of i, $i > 0$, each of the following two sets of payments has present value K:
   (i) A payment of 121 immediately and another payment of 121 at the end of one year.
   (ii) A payment of 144 at the end of two years and another payment of 144 at the end of three years.

   Calculate K. [SOA 11/92 #2]

   (A) 237   (B) 232   (C) 227   (D) 222   (E) 217

4. A business permits its customers to pay with a credit card or to receive a percentage discount r for paying cash.

   For credit card purchases, the business receives 95% of the purchase price one-half month later.

   At an effective annual rate of 12%, the two payment methods are equivalent.

   Determine r. [SOA 11/92 #3]

   (A) 4.55   (B) 4.85   (C) 5.15   (D) 5.45   (E) 5.75

5. At an annual effective interest rate of i, $i > 0$, the following are all equal:
   (i) the present value of 10000 at the end of 6 years;
   (ii) the sum of the present values of 6000 at the end of year t and 56000 at the end of year $2t$; and
   (iii) 5000 immediately.

   Calculate the present value of a payment of 8000 at the end of year $t + 3$ using the same annual effective interest rate. [SOA 11/90 #4]

   (A) 1330   (B) 1414   (C) 1600   (D) 1775   (E) 2000

6. An investor puts 100 into Fund X and 100 into Fund Y. Fund Y earns compound interest at the annual rate of $j > 0$, and Fund X earns simple interest at the annual rate of $1.05j$.

   At the end of 2 years, the amount in Fund Y is equal to the amount in Fund X.

   Calculate the amount in Fund Y at the end of 5 years. [SOA 5/90 #1]

   (A) 150   (B) 153   (C) 157   (D) 161   (E) 165

7. (This question should actually appear after Section 1a(vi) to 1a(ix).) You are given:
   (i) Fund X accumulates at an interest rate of 8% compounded quarterly;
   (ii) Fund Y accumulates at an interest rate of 6% compounded semiannually;
   (iii) at the end of 10 years, the total amount in the two funds combined is 1000; and
   (iv) at the end of 5 years, the amount in Fund X is twice that in Fund Y.

Calculate the total amount in the two funds at the end of 2 years. [SOA 5/90 #4]

(A) 560   (B) 570   (C) 580   (D) 590   (E) 600

8. Gertrude deposits 10,000 in a bank. During the first year, the bank credits an annual effective rate of interest i. During the second year, the bank credits an annual effective rate of interest $(i - 5\%)$. At the end of two years, she has 12,093.75 in the bank.

   What would Gertrude have in the bank at the end of three years, if the annual effective rate of interest were $(i + 9\%)$ for each of the three years? [SOA 11/83 #3]

   (A) 16,851   (B) 17,196   (C) 17,499   (D) 17,936   (E) 18,113

9. John borrows 1,000 from Jane at an annual effective rate of interest i. He agrees to pay back 1,000 after six years and 1,366.87 after another six years.

   Three years after his first payment, John repays the outstanding balance.

   What is the amount of John's second payment? [SOA 11/88 #4]

   (A) 1,020   (B) 1,027   (C) 1,048   (D) 1,073   (E) 1,094

10. Two funds, A and B, start with the same amount.

    Fund A grows at an annual interest rate of $i > 0$ for n years, and at an annual interest rate of $j > 0$ for the next n years.

    Fund B grows at an annual interest rate of $k > 0$ for $2n$ years.

    Fund A equals 1.5 times fund B after n years. The amount in the two funds are equal after $2n$ years.

    Which of the following are true?

        (i) $j < k < i$
       (ii) $k < \frac{i+j}{2}$
      (iii) $j = k \left(\frac{2}{3}\right)^{1/n}$

    [SOA 5/88 #3]

    (A) I and II only   (B) I and III only   (C) II and III only   (D) I, II and III
    (E) The correct answer is not given by (A), (B), (C), or (D).

11. Fund A is invested at an effective annual interest rate of 3%.

    Fund B is invested at an effective annual interest rate of 2.5%.

    At the end of 20 years, the total in the two funds is 10,000. At the end of 31 years, the amount in Fund A is twice the amount in Fund B.

    Calculate the total in the two funds at the end of 10 years. [SOA 11/87 #18]

    (A) 5,732   (B) 6,602   (C) 7,472   (D) 7,569   (E) 8,123

12. You are given:
       (i) The sum of the present values of a payment of X at the end of 10 years and a payment of Y at the end of 20 years is equal to the present value of a payment of $X + Y$ at the end of 15 years.
       (ii) $X + Y = 100$
      (iii) $i = 5\%$
    Calculate X. [SOA 5/87 #8]

    (A) 44   (B) 48   (C) 50   (D) 52   (E) 57

13. The present value of 200 paid at the end of n years, plus the present value of 100 paid at the end of $2n$ years is 200.

    Determine the annual effective rate of interest. [SOA 11/85 #3]

(A) $\left(\frac{\sqrt{3}+1}{2}\right)^{1/n} - 1$

(B) $1 - \left(\frac{\sqrt{3}-1}{2}\right)^{1/n}$

(C) $\left(\frac{\sqrt{3}-1}{2}\right)^{1/n} - 1$

(D) $\left(\frac{1+\sqrt{3}}{2}\right)^{1/2n} - 1$

(E) $1 - \left(\frac{\sqrt{3}-1}{2}\right)^{1/2n}$