Contents

Lesson 7

Survival Distributions: Fractional Ages

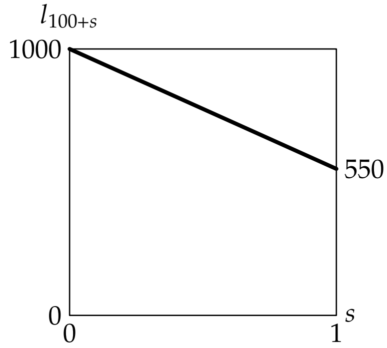
Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 3.2

Life tables list mortality rates (q_x) or lives (l_x) for integral ages only. Often, it is necessary to determine lives at fractional ages (like $l_{x+0.5}$ for x an integer) or mortality rates for fractions of a year. We need some way to interpolate between ages.

7.1 Uniform distribution of deaths

The easiest interpolation method is linear interpolation, or uniform distribution of deaths between integral ages (UDD). This means that the number of lives at age $x+s$, $0 \le s \le 1$, is a weighted average of the number of lives at age x and the number of lives at age $x+1$:

$$l_{x+s} = (1-s)l_x + sl_{x+1} = l_x - sd_x \tag{7.1}$$



The graph of l_{x+s} is a straight line between $s=0$ and $s=1$ with slope $-d_x$. The graph at the right portrays this for a mortality rate $q_{100} = 0.45$ and $l_{100} = 1000$.

Contrast UDD with an assumption of a uniform survival function. If age at death is uniformly distributed, then l_x as a function of x is a straight line. If UDD is assumed, l_x is a straight line between integral ages, but the slope may vary for different ages. Thus if age at death is uniformly distributed, UDD holds at all ages, but not conversely.

Using l_{x+s}, we can compute ${}_sq_x$:

$$\begin{aligned}{}_sq_x &= 1 - {}_sp_x \\ &= 1 - \frac{l_{x+s}}{l_x} = 1 - (1 - sq_x) = sq_x\end{aligned} \tag{7.2}$$

That is one of the most important formulas, so let's state it again:

$$\boxed{{}_sq_x = sq_x} \tag{7.2}$$

More generally, for $0 \le s+t \le 1$,

$$\begin{aligned}{}_sq_{x+t} &= 1 - {}_sp_{x+t} = 1 - \frac{l_{x+s+t}}{l_{x+t}} \\ &= 1 - \frac{l_x - (s+t)d_x}{l_x - td_x} = \frac{sd_x}{l_x - td_x} = \frac{sq_x}{1 - tq_x}\end{aligned} \tag{7.3}$$

where the last equation was obtained by dividing numerator and denominator by l_x. The important point to pick up is that while ${}_sq_x$ is the proportion of the year s times q_x, the corresponding concept at age $x+t$, ${}_sq_{x+t}$, is *not* sq_x, but is in fact higher than sq_x. The *number* of lives dying in any amount of time is constant, and since there are fewer and fewer lives as the year progresses, the *rate* of death is in fact increasing over the year. The numerator of ${}_sq_{x+t}$ is the proportion of the year being measured s times the death rate, but then this must be divided by 1 minus the proportion of the year that elapsed before the start of measurement.

For most problems involving death probabilities, it will suffice if you remember that l_{x+s} is linearly interpolated. It often helps to create a life table with an arbitrary radix. Try working out the following example before looking at the answer.

Example 7A You are given:

(i) $q_x = 0.1$

(ii) Uniform distribution of deaths between integral ages is assumed.

Calculate ${}_{1/2}q_{x+1/4}$.

Answer: Let $l_x = 1$. Then $l_{x+1} = l_x(1 - q_x) = 0.9$ and $d_x = 0.1$. Linearly interpolating,

$$\begin{aligned} l_{x+1/4} &= l_x - \tfrac{1}{4}d_x = 1 - \tfrac{1}{4}(0.1) = 0.975 \\ l_{x+3/4} &= l_x - \tfrac{3}{4}d_x = 1 - \tfrac{3}{4}(0.1) = 0.925 \\ {}_{1/2}q_{x+1/4} &= \frac{l_{x+1/4} - l_{x+3/4}}{l_{x+1/4}} = \frac{0.975 - 0.925}{0.975} = \boxed{\mathbf{0.051282}} \end{aligned}$$

You could also use equation (7.3) to work this example. □

Example 7B For two lives age (x) with independent future lifetimes, ${}_{k|}q_x = 0.1(k + 1)$ for $k = 0, 1, 2$. Deaths are uniformly distributed between integral ages.

Calculate the probability that both lives will survive 2.25 years.

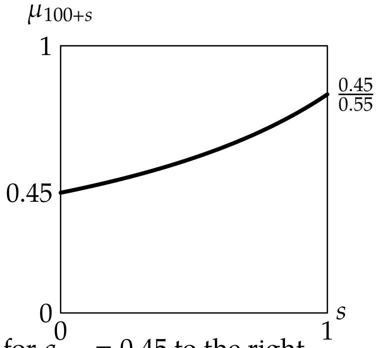
Answer: Since the two lives are independent, the probability of both surviving 2.25 years is the square of ${}_{2.25}p_x$, the probability of one surviving 2.25 years. If we let $l_x = 1$ and use $d_{x+k} = l_x\,{}_{k|}q_x$, we get

$$\begin{aligned} q_x &= 0.1(1) = 0.1 & l_{x+1} &= 1 - d_x = 1 - 0.1 = 0.9 \\ {}_{1|}q_x &= 0.1(2) = 0.2 & l_{x+2} &= 0.9 - d_{x+1} = 0.9 - 0.2 = 0.7 \\ {}_{2|}q_x &= 0.1(3) = 0.3 & l_{x+3} &= 0.7 - d_{x+2} = 0.7 - 0.3 = 0.4 \end{aligned}$$

Then linearly interpolating between l_{x+2} and l_{x+3}, we get

$$\begin{aligned} l_{x+2.25} &= 0.7 - 0.25(0.3) = 0.625 \\ {}_{2.25}p_x &= \frac{l_{x+2.25}}{l_x} = 0.625 \end{aligned}$$

Squaring, the answer is $0.625^2 = \boxed{\mathbf{0.390625}}$. □



The probability density function of T_x, ${}_sp_x\,\mu_{x+s}$, is the constant q_x, the derivative of the conditional cumulative distribution function ${}_sq_x = sq_x$ with respect to s. That is another important formula, since the density is needed to compute expected values, so let's repeat it:

$$\boxed{{}_sp_x\,\mu_{x+s} = q_x} \tag{7.4}$$

It follows that the force of mortality is q_x divided by $1 - sq_x$:

$$\mu_{x+s} = \frac{q_x}{{}_sp_x} = \frac{q_x}{1 - sq_x} \tag{7.5}$$

The force of mortality increases over the year, as illustrated in the graph for $q_{100} = 0.45$ to the right.

Quiz 7-1 You are given:

(i) $\mu_{50.4} = 0.01$

(ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.6}q_{50.4}$.

Complete Expectation of Life Under UDD

Under uniform distribution of deaths between integral ages, if the complete future lifetime random variable T_x is written as $T_x = K_x + R_x$, where K_x is the curtate future lifetime and R_x is the fraction of the last year lived, then K_x and R_x are independent, and R_x is uniform on $[0,1)$. If uniform distribution of deaths is not assumed, K_x and R_x are usually not independent. Since R_x is uniform on $[0,1)$, $\mathbf{E}[R_x] = \frac{1}{2}$ and $\operatorname{Var}(R_x) = \frac{1}{12}$. It follows from $\mathbf{E}[R_x] = \frac{1}{2}$ that

$$\mathring{e}_x = e_x + \tfrac{1}{2} \tag{7.6}$$

Let's discuss temporary complete life expectancy. You can always evaluate the temporary complete expectancy, whether or not UDD is assumed, by integrating ${}_tp_x$, as indicated by formula (5.6) on page 80. For UDD, ${}_tp_x$ is linear between integral ages. Therefore, a rule we learned in Lesson 5 applies for all integral x:

$$\mathring{e}_{x:\overline{1}|} = p_x + 0.5q_x \tag{5.13}$$

This equation will be useful. In addition, the method for generating this equation can be used to work out questions involving temporary complete life expectancies for short periods. The following example illustrates this. This example will be reminiscent of calculating temporary complete life expectancy for uniform mortality.

Example 7C You are given

(i) $q_x = 0.1$.

(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x:\overline{0.4}|}$.

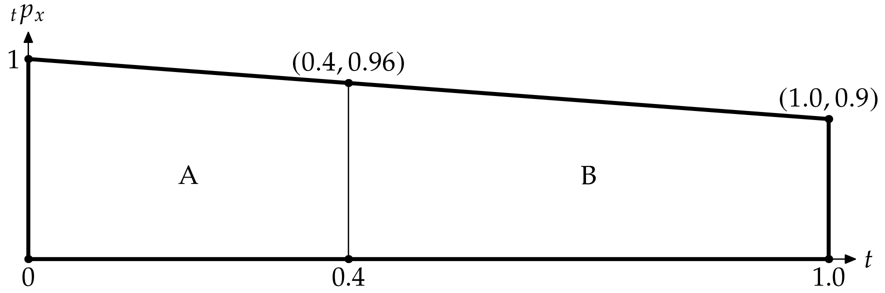
Answer: We will discuss two ways to solve this: an algebraic method and a geometric method.

The algebraic method is based on the double expectation theorem, equation (1.11). It uses the fact that *for a uniform distribution, the mean is the midpoint*. If deaths occur uniformly between integral ages, then those who die within a period contained within a year survive half the period on the average.

In this example, those who die within 0.4 survive an average of 0.2. Those who survive 0.4 survive an average of 0.4 of course. The temporary life expectancy is the weighted average of these two groups, or ${}_{0.4}q_x(0.2) + {}_{0.4}p_x(0.4)$. This is:

$$\begin{aligned}
{}_{0.4}q_x &= (0.4)(0.1) = 0.04\\
{}_{0.4}p_x &= 1 - 0.04 = 0.96\\
\mathring{e}_{x:\overline{0.4}|} &= 0.04(0.2) + 0.96(0.4) = \boxed{0.392}
\end{aligned}$$

An equivalent geometric method, the trapezoidal rule, is to draw the ${}_tp_x$ function from 0 to 0.4. The integral of ${}_tp_x$ is the area under the line, which is the area of a trapezoid: the average of the heights times the width. The following is the graph (not drawn to scale):



Trapezoid A is the area we are interested in. Its area is $\frac{1}{2}(1+0.96)(0.4)=\boxed{0.392}$. ☐

Quiz 7-2 As in Example 7C, you are given

(i) $q_x = 0.1$.
(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.4:\overline{0.6|}}$.

Let's now work out an example in which the duration crosses an integral boundary.

Example 7D You are given:

(i) $q_x = 0.1$
(ii) $q_{x+1} = 0.2$
(iii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.5:\overline{1|}}$.

Answer: Let's start with the algebraic method. Since the mortality rate changes at $x+1$, we must split the group into those who die before $x+1$, those who die afterwards, and those who survive. Those who die before $x+1$ live 0.25 on the average since the period to $x+1$ is length 0.5. Those who die after $x+1$ live between 0.5 and 1 years; the midpoint of 0.5 and 1 is 0.75, so they live 0.75 years on the average. Those who survive live 1 year.

Now let's calculate the probabilities.

$$\begin{aligned}
{}_{0.5}q_{x+0.5} &= \frac{0.5(0.1)}{1-0.5(0.1)} = \frac{5}{95}\\
{}_{0.5}p_{x+0.5} &= 1-\frac{5}{95} = \frac{90}{95}\\
{}_{0.5|0.5}q_{x+0.5} &= \left(\frac{90}{95}\right)\bigl(0.5(0.2)\bigr) = \frac{9}{95}\\
{}_{1}p_{x+0.5} &= 1-\frac{5}{95}-\frac{9}{95} = \frac{81}{95}
\end{aligned}$$

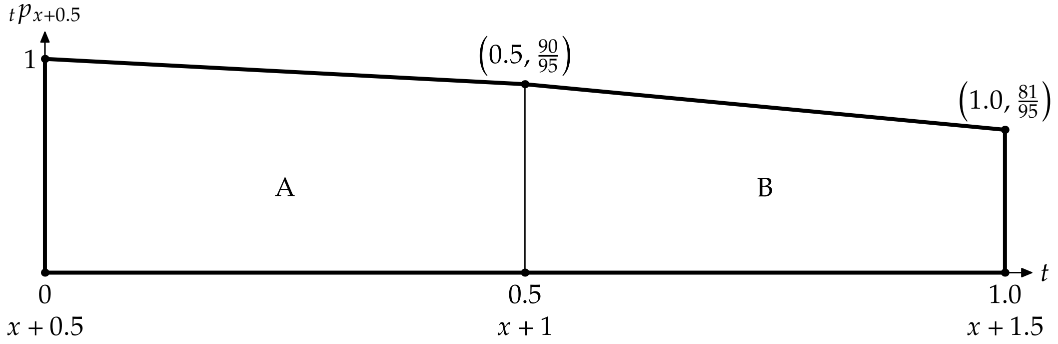
These probabilities could also be calculated by setting up an l_x table with radix 100 at age x and interpolating within it to get $l_{x+0.5}$ and $l_{x+1.5}$. Then

$$\begin{aligned}
l_{x+1} &= 0.9 l_x = 90\\
l_{x+2} &= 0.8 l_{x+1} = 72\\
l_{x+0.5} &= 0.5(90+100) = 95\\
l_{x+1.5} &= 0.5(72+90) = 81\\
{}_{0.5}q_{x+0.5} &= 1 - \frac{90}{95} = \frac{5}{95}\\
{}_{0.5|0.5}q_{x+0.5} &= \frac{90-81}{95} = \frac{9}{95}\\
{}_{1}p_{x+0.5} &= \frac{l_{x+1.5}}{l_{x+0.5}} = \frac{81}{95}
\end{aligned}$$

Either way, we're now ready to calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

$$\mathring{e}_{x+0.5:\overline{1}|} = \frac{5(0.25) + 9(0.75) + 81(1)}{95} = \boxed{\mathbf{\frac{89}{95}}}$$

For the geometric method we draw the following graph:



The heights at $x+1$ and $x+1.5$ are as we computed above. Then we compute each area separately. The area of A is $\frac{1}{2}\left(1+\frac{90}{95}\right)(0.5) = \frac{185}{95(4)}$. The area of B is $\frac{1}{2}\left(\frac{90}{95}+\frac{81}{95}\right)(0.5) = \frac{171}{95(4)}$. Adding them up, we get $\frac{185+171}{95(4)} = \boxed{\mathbf{\frac{89}{95}}}$. □

Quiz 7-3 The probability that a battery fails by the end of the k^{th} month is given in the following table:

k	Probability of battery failure by the end of month k
1	0.05
2	0.20
3	0.60

Between integral months, time of failure for the battery is uniformly distributed.
Calculate the expected amount of time the battery survives within 2.25 months.

To calculate $\mathring{e}_{x:\overline{n}|}$ in terms of $e_{x:\overline{n}|}$ when x and n are both integers, note that those who survive n years contribute the same to both. Those who die contribute an average of $\frac{1}{2}$ more to $\mathring{e}_{x:\overline{n}|}$ since they die on the average in the middle of the year. Thus the difference is $\frac{1}{2}{}_nq_x$:

$$\mathring{e}_{x:\overline{n|}} = e_{x:\overline{n|}} + 0.5{}_nq_x \tag{7.7}$$

Example 7E You are given:

(i) $q_x = 0.01$ for $x = 50, 51, \ldots, 59$.
(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{50:\overline{10|}}$.

Answer: As we just said, $\mathring{e}_{50:\overline{10|}} = e_{50:\overline{10|}} + 0.5{}_{10}q_{50}$. The first summand, $e_{50:\overline{10|}}$, is the sum of ${}_kp_{50} = 0.99^k$ for $k = 1, \ldots, 10$. This sum is a geometric series:

$$e_{50:\overline{10|}} = \sum_{k=1}^{10} 0.99^k = \frac{0.99 - 0.99^{11}}{1 - 0.99} = 9.46617$$

The second summand, the probability of dying within 10 years is ${}_{10}q_{50} = 1 - 0.99^{10} = 0.095618$. Therefore

$$\mathring{e}_{50:\overline{10|}} = 9.46617 + 0.5(0.095618) = \boxed{\mathbf{9.51398}}$$ □

7.2 Constant force of mortality

The constant force of mortality interpolation method sets μ_{x+s} equal to a constant for x an integral age and $0 < s \le 1$. Since $p_x = \exp\left(-\int_0^1 \mu_{x+s}\,\mathrm{d}s\right)$ and $\mu_{x+s} = \mu$ is constant,

$$p_x = e^{-\mu} \tag{7.8}$$

$$\mu = -\ln p_x \tag{7.9}$$

Therefore

$${}_sp_x = e^{-\mu s} = (p_x)^s \tag{7.10}$$

In fact, ${}_sp_{x+t}$ is independent of t for $0 \le t \le 1 - s$.

$${}_sp_{x+t} = (p_x)^s \tag{7.11}$$

for any $0 \le t \le 1 - s$. Figure 7.1 shows l_{100+s} and μ_{100+s} for $l_{100} = 1000$ and $q_{100} = 0.45$ if constant force of mortality is assumed.

Contrast constant force of mortality between integral ages to global constant force of mortality, which was introduced in Subsection 4.2.1. The method discussed here allows μ_x to vary for different integers x.

We will now repeat some of the earlier examples but using constant force of mortality.

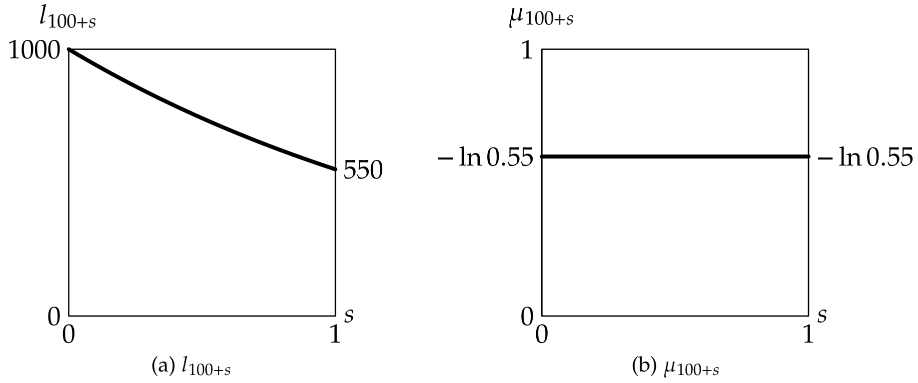
Example 7F You are given:

(i) $q_x = 0.1$
(ii) The force of mortality is constant between integral ages.

Calculate ${}_{1/2}q_{x+1/4}$.

Answer:

$${}_{1/2}q_{x+1/4} = 1 - {}_{1/2}p_{x+1/4} = 1 - p_x^{1/2} = 1 - 0.9^{1/2} = 1 - 0.948683 = \boxed{\mathbf{0.051317}}$$ □



(a) l_{100+s} (b) μ_{100+s}

Figure 7.1: Example of constant force of mortality

Example 7G You are given:

(i) $q_x = 0.1$
(ii) $q_{x+1} = 0.2$
(iii) The force of mortality is constant between integral ages.

Calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

Answer: We calculate $\int_0^1 {}_tp_{x+0.5}\,\mathrm{d}t$. We split this up into two integrals, one from 0 to 0.5 for age x and one from 0.5 to 1 for age $x+1$. The first integral is

$$\int_0^{0.5} {}_tp_{x+0.5}\,\mathrm{d}t = \int_0^{0.5} p_x^t\,\mathrm{d}t = \int_0^{0.5} 0.9^t\,\mathrm{d}t = -\frac{1-0.9^{0.5}}{\ln 0.9} = 0.487058$$

For $t > 0.5$,

$$ {}_tp_{x+0.5} = {}_{0.5}p_{x+0.5}\,{}_{t-0.5}p_{x+1} = 0.9^{0.5}\,{}_{t-0.5}p_{x+1}$$

so the second integral is

$$0.9^{0.5}\int_{0.5}^{1} {}_{t-0.5}p_{x+1}\,\mathrm{d}t = 0.9^{0.5}\int_0^{0.5} 0.8^t\,\mathrm{d}t = -\left(0.9^{0.5}\right)\left(\frac{1-0.8^{0.5}}{\ln 0.8}\right) = (0.948683)(0.473116) = 0.448837$$

The answer is $\mathring{e}_{x+0.5:\overline{1}|} = 0.487058 + 0.448837 = \boxed{\mathbf{0.935895}}$. □

Although constant force of mortality is not used as often as UDD, it can be useful for simplifying formulas under certain circumstances. Calculating the expected present value of an insurance where the death benefit within a year follows an exponential pattern (this can happen when the death benefit is the discounted present value of something) may be easier with constant force of mortality than with UDD.

The formulas for this lesson are summarized in Table 7.1.

Table 7.1: Summary of formulas for fractional ages

Function	Uniform distribution of deaths	Constant force of mortality
l_{x+s}	$l_x - sd_x$	$l_x\, p_x^s$
${}_sq_x$	sq_x	$1 - p_x^s$
${}_sp_x$	$1 - sq_x$	p_x^s
${}_sq_{x+t}$	$sq_x/(1 - tq_x)$	$1 - p_x^s$
μ_{x+s}	$q_x/(1 - sq_x)$	$-\ln p_x$
${}_sp_x\, \mu_{x+s}$	q_x	$-p_x^s \ln p_x$
$\mathring{e}_x$	$e_x + 0.5$	
$\mathring{e}_{x:\overline{n}\rceil}$	$e_{x:\overline{n}\rceil} + 0.5\, {}_nq_x$	
$\mathring{e}_{x:\overline{1}\rceil}$	$p_x + 0.5q_x$	

Exercises

Uniform distribution of death

7.1. [**CAS4-S85:16**] (1 point) Deaths are uniformly distributed between integral ages.

Which of the following represents ${}_{3/4}p_x + \frac{1}{2}{}_{1/2}p_x\, \mu_{x+1/2}$?

(A) ${}_{3/4}p_x$ (B) ${}_{3/4}q_x$ (C) ${}_{1/2}p_x$ (D) ${}_{1/2}q_x$ (E) ${}_{1/4}p_x$

7.2. [Based on **150-S88:25**] You are given:

(i) ${}_{0.25}q_{x+0.75} = 3/31$.
(ii) Mortality is uniformly distributed within age x.

Calculate q_x.

Use the following information for questions 7.3 and 7.4:

You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_x = 0.10$.
(iii) $q_{x+1} = 0.15$.

7.3. Calculate ${}_{1/2}q_{x+3/4}$.

7.4. Calculate ${}_{0.3|0.5}q_{x+0.4}$.

Exercises continue on the next page . . .

7.5. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) Mortality follows the Illustrative Life Table.

Calculate the median future lifetime for (45.5).

7.6. [**160-F90:5**] You are given:

(i) A survival distribution is defined by

$$l_x = 1000\left(1 - \left(\frac{x}{100}\right)^2\right),\ 0 \le x \le 100.$$

(ii) μ_x denotes the actual force of mortality for the survival distribution.
(iii) μ_x^L denotes the approximation of the force of mortality based on the uniform distribution of deaths assumption for l_x, $50 \le x < 51$.

Calculate $\mu_{50.25} - \mu_{50.25}^L$.

(A) −0.00016 (B) −0.00007 (C) 0 (D) 0.00007 (E) 0.00016

7.7. A survival distribution is defined by

(i) $S_0(k) = 1/(1 + 0.01k)^4$ for k a non-negative integer.
(ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.4}q_{20.2}$.

7.8. [Based on **150-S89:15**] You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii)

x	l_x
35	100
36	99
37	96
38	92
39	87

Which of the following are true?

I. ${}_{1|2}q_{36} = 0.091$
II. $\mu_{37.5} = 0.043$
III. ${}_{0.33}q_{38.5} = 0.021$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

7.9. **[150-82-94:5]** You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) ${}_{0.75}p_x = 0.25$.

Which of the following are true?

I. ${}_{0.25}q_{x+0.5} = 0.5$
II. ${}_{0.5}q_x = 0.5$
III. $\mu_{x+0.5} = 0.5$

(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

7.10. **[3-S00:12]** For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$
(ii) $\mu_{81.5} = 0.0408$
(iii) $\mu_{82.5} = 0.0619$
(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

7.11. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_x = 0.1$.
(iii) $q_{x+1} = 0.3$.

Calculate $\mathring{e}_{x+0.7:\overline{1}|}$.

7.12. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_{45} = 0.01$.
(iii) $q_{46} = 0.011$.

Calculate $\operatorname{Var}\Big(\min(T_{45}, 2)\Big)$.

7.13. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) ${}_{10}p_x = 0.2$.

Calculate $\mathring{e}_{x:\overline{10}|} - e_{x:\overline{10}|}$.

Exercises continue on the next page . . .

7.14. **[4-F86:21]** You are given:

(i) $q_{60} = 0.020$
(ii) $q_{61} = 0.022$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{60:\overline{1.5}|}$.

(A) 1.447 (B) 1.457 (C) 1.467 (D) 1.477 (E) 1.487

7.15. **[150-F89:21]** You are given:

(i) $q_{70} = 0.040$
(ii) $q_{71} = 0.044$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{70:\overline{1.5}|}$.

(A) 1.435 (B) 1.445 (C) 1.455 (D) 1.465 (E) 1.475

7.16. **[3-S01:33]** For a 4-year college, you are given the following probabilities for dropout from all causes:

$$\begin{aligned} q_0 &= 0.15 \\ q_1 &= 0.10 \\ q_2 &= 0.05 \\ q_3 &= 0.01 \end{aligned}$$

Dropouts are uniformly distributed over each year.

Compute the temporary 1.5-year complete expected college lifetime of a student entering the second year, $\mathring{e}_{1:\overline{1.5}|}$.

(A) 1.25 (B) 1.30 (C) 1.35 (D) 1.40 (E) 1.45

7.17. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $\mathring{e}_{x+0.5:\overline{0.5}|} = 5/12$.

Calculate q_x.

7.18. You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) $\mathring{e}_{55.2:\overline{0.4}|} = 0.396$.

Calculate $\mu_{55.2}$.

7.19. **[150-S87:21]** You are given:

(i) $d_x = k$ for $x = 0, 1, 2, \ldots, \omega - 1$
(ii) $\mathring{e}_{20:\overline{20}|} = 18$
(iii) Deaths are uniformly distributed over each year of age.

Calculate ${}_{30|10}q_{30}$.

(A) 0.111 (B) 0.125 (C) 0.143 (D) 0.167 (E) 0.200

Exercises continue on the next page . . .

7.20. **[150-S89:24]** You are given:

(i) Deaths are uniformly distributed over each year of age.
(ii) $\mu_{45.5} = 0.5$

Calculate $\mathring{e}_{45:\overline{1}|}$.

(A) 0.4 (B) 0.5 (C) 0.6 (D) 0.7 (E) 0.8

7.21. **[CAS3-S04:10]** 4,000 people age (30) each pay an amount, P, into a fund. Immediately after the 1,000th death, the fund will be dissolved and each of the survivors will be paid \$50,000.

- Mortality follows the Illustrative Life Table, using linear interpolation at fractional ages.
- $i = 12\%$

Calculate P.

(A) Less than 515
(B) At least 515, but less than 525
(C) At least 525, but less than 535
(D) At least 535, but less than 545
(E) At least 545

Constant force of mortality

7.22. **[160-F87:5]** Based on given values of l_x and l_{x+1}, ${}_{1/4}p_{x+1/4} = 49/50$ under the assumption of constant force of mortality.

Calculate ${}_{1/4}p_{x+1/4}$ under the uniform distribution of deaths hypothesis.

(A) 0.9799 (B) 0.9800 (C) 0.9801 (D) 0.9802 (E) 0.9803

7.23. **[160-S89:5]** A mortality study is conducted for the age interval $(x, x+1]$.

If a constant force of mortality applies over the interval, ${}_{0.25}q_{x+0.1} = 0.05$.

Calculate ${}_{0.25}q_{x+0.1}$ assuming a uniform distribution of deaths applies over the interval.

(A) 0.044 (B) 0.047 (C) 0.050 (D) 0.053 (E) 0.056

7.24. **[150-F89:29]** You are given that $q_x = 0.25$.

Based on the constant force of mortality assumption, the force of mortality is μ^A_{x+s}, $0 < s < 1$.

Based on the uniform distribution of deaths assumption, the force of mortality is μ^B_{x+s}, $0 < s < 1$.

Calculate the smallest s such that $\mu^B_{x+s} \geq \mu^A_{x+s}$.

(A) 0.4523 (B) 0.4758 (C) 0.5001 (D) 0.5239 (E) 0.5477

Exercises continue on the next page . . .

7.25. **[160-S91:4]** From a population mortality study, you are given:

(i) Within each age interval, $[x+k, x+k+1)$, the force of mortality, μ_{x+k}, is constant.

(ii)

k	$e^{-\mu_{x+k}}$	$\dfrac{1-e^{-\mu_{x+k}}}{\mu_{x+k}}$
0	0.98	0.99
1	0.96	0.98

Calculate $\mathring{e}_{x:\overline{2}|}$, the expected lifetime in years over $(x, x+2]$.

(A) 1.92 (B) 1.94 (C) 1.95 (D) 1.96 (E) 1.97

7.26. You are given:

(i) $q_{80} = 0.1$
(ii) $q_{81} = 0.2$
(iii) The force of mortality is constant between integral ages.

Calculate $\mathring{e}_{80.4:\overline{0.8}|}$.

7.27. **[3-S01:27]** An actuary is modeling the mortality of a group of 1000 people, each age 95, for the next three years.

The actuary starts by calculating the expected number of survivors at each integral age by

$$l_{95+k} = 1000\,{}_kp_{95}, \qquad k = 1, 2, 3$$

The actuary subsequently calculates the expected number of survivors at the middle of each year using the assumption that deaths are uniformly distributed over each year of age.

This is the result of the actuary's model:

Age	Survivors
95	1000
95.5	800
96	600
96.5	480
97	—
97.5	288
98	—

The actuary decides to change his assumption for mortality at fractional ages to the constant force assumption. He retains his original assumption for each ${}_kp_{95}$.

Calculate the revised expected number of survivors at age 97.5.

(A) 270 (B) 273 (C) 276 (D) 279 (E) 282

7.28. [**M-F06:16**] You are given the following information on participants entering a 2-year program for treatment of a disease:

(i) Only 10% survive to the end of the second year.
(ii) The force of mortality is constant within each year.
(iii) The force of mortality for year 2 is three times the force of mortality for year 1.

Calculate the probability that a participant who survives to the end of month 3 dies by the end of month 21.

(A) 0.61 (B) 0.66 (C) 0.71 (D) 0.75 (E) 0.82

7.29. [**Sample Question #267**] You are given:

(i) $\mu_x = \sqrt{\dfrac{1}{80-x}}, \quad 0 \le x \le 80$
(ii) F is the exact value of $S_0(10.5)$.
(iii) G is the value of $S_0(10.5)$ using the constant force assumption for interpolation between ages 10 and 11.

Calculate $F - G$.

(A) −0.01083 (B) −0.00005 (C) 0 (D) 0.00003 (E) 0.00172

Additional old SOA Exam MLC questions: S12:2, F13:25, F16:1

Additional old CAS Exam 3/3L questions: S05:31, F05:13, S06:13, F06:13, S07:24, S08:16, S09:3, F09:3, S10:4, F10:3, S11:3, S12:3, F12:3, S13:3, F13:3

Additional old CAS Exam LC questions: S14:4, F14:4, S15:3, F15:3

Solutions

7.1. In the second summand, ${}_{1/2}p_x\,\mu_{x+1/2}$ is the density function, which is the constant q_x under UDD. The first summand ${}_{3/4}p_x = 1 - \frac{3}{4}q_x$. So the sum is $1 - \frac{1}{4}q_x$, or $\boxed{{}_{1/4}p_x}$. **(E)**

7.2. Using equation (7.3),

$$\begin{aligned}\frac{3}{31} &= {}_{0.25}q_{x+0.75} = \frac{0.25q_x}{1-0.75q_x}\\ \frac{3}{31} - \frac{2.25}{31}q_x &= 0.25q_x\\ \frac{3}{31} &= \frac{10}{31}q_x\\ q_x &= \boxed{0.3}\end{aligned}$$

7.3. We calculate the probability that $(x+\frac{3}{4})$ survives for half a year. Since the duration crosses an integer boundary, we break the period up into two quarters of a year. The probability of $(x+3/4)$ surviving for 0.25 years is, by equation (7.3),

$${}_{1/4}p_{x+3/4} = \frac{1-0.10}{1-0.75(0.10)} = \frac{0.9}{0.925}$$

The probability of $(x+1)$ surviving to $x+1.25$ is

$$_{1/4}p_{x+1} = 1 - 0.25(0.15) = 0.9625$$

The answer to the question is then the complement of the product of these two numbers:

$$_{1/2}q_{x+3/4} = 1 - {}_{1/2}p_{x+3/4} = 1 - {}_{1/4}p_{x+3/4}\,{}_{1/4}p_{x+1} = 1 - \left(\frac{0.9}{0.925}\right)(0.9625) = \boxed{\mathbf{0.06351}}$$

Alternatively, you could build a life table starting at age x, with $l_x = 1$. Then $l_{x+1} = (1-0.1) = 0.9$ and $l_{x+2} = 0.9(1-0.15) = 0.765$. Under UDD, l_x at fractional ages is obtained by linear interpolation, so

$$\begin{aligned} l_{x+0.75} &= 0.75(0.9) + 0.25(1) = 0.925 \\ l_{x+1.25} &= 0.25(0.765) + 0.75(0.9) = 0.86625 \\ _{1/2}p_{3/4} &= \frac{l_{x+1.25}}{l_{x+0.75}} = \frac{0.86625}{0.925} = 0.93649 \\ _{1/2}q_{3/4} &= 1 - {}_{1/2}p_{3/4} = 1 - 0.93649 = \boxed{\mathbf{0.06351}} \end{aligned}$$

7.4. $_{0.3|0.5}q_{x+0.4}$ is $_{0.3}p_{x+0.4} - {}_{0.8}p_{x+0.4}$. The first summand is

$$_{0.3}p_{x+0.4} = \frac{1-0.7q_x}{1-0.4q_x} = \frac{1-0.07}{1-0.04} = \frac{93}{96}$$

The probability that $(x+0.4)$ survives to $x+1$ is, by equation (7.3),

$$_{0.6}p_{x+0.4} = \frac{1-0.10}{1-0.04} = \frac{90}{96}$$

and the probability $(x+1)$ survives to $x+1.2$ is

$$_{0.2}p_{x+1} = 1 - 0.2q_{x+1} = 1 - 0.2(0.15) = 0.97$$

So

$$_{0.3|0.5}q_{x+0.4} = \frac{93}{96} - \left(\frac{90}{96}\right)(0.97) = \boxed{\mathbf{0.059375}}$$

Alternatively, you could use the life table from the solution to the last question, and linearly interpolate:

$$\begin{aligned} l_{x+0.4} &= 0.4(0.9) + 0.6(1) = 0.96 \\ l_{x+0.7} &= 0.7(0.9) + 0.3(1) = 0.93 \\ l_{x+1.2} &= 0.2(0.765) + 0.8(0.9) = 0.873 \\ _{0.3|0.5}q_{x+0.4} &= \frac{0.93 - 0.873}{0.96} = \boxed{\mathbf{0.059375}} \end{aligned}$$

7.5. Under uniform distribution of deaths between integral ages, $l_{x+0.5} = \frac{1}{2}(l_x + l_{x+1})$, since the survival function is a straight line between two integral ages. Therefore, $l_{45.5} = \frac{1}{2}(9{,}164{,}051 + 9{,}127{,}426) = 9{,}145{,}738.5$. Median future lifetime occurs when $l_x = \frac{1}{2}(9{,}145{,}738.5) = 4{,}572{,}869$. This happens between ages 77 and 78. We interpolate between the ages to get the exact median:

$$\begin{aligned} l_{77} - s(l_{77} - l_{78}) &= 4{,}572{,}869 \\ 4{,}828{,}182 - s(4{,}828{,}182 - 4{,}530{,}360) &= 4{,}572{,}869 \\ 4{,}828{,}182 - 297{,}822s &= 4{,}572{,}869 \\ s = \frac{4{,}828{,}182 - 4{,}572{,}869}{297{,}822} = \frac{255{,}313}{297{,}822} &= 0.8573 \end{aligned}$$

So the median age at death is 77.8573, and median future lifetime is $77.8573 - 45.5 = \boxed{\mathbf{32.3573}}$.

7.6. ${}_xp_0 = \frac{l_x}{l_0} = 1 - \left(\frac{x}{100}\right)^2$. The force of mortality is calculated as the negative derivative of $\ln {}_xp_0$:

$$\mu_x = -\frac{\mathrm{d}\ln {}_xp_0}{\mathrm{d}x} = \frac{2\left(\frac{x}{100}\right)\left(\frac{1}{100}\right)}{1-\left(\frac{x}{100}\right)^2} = \frac{2x}{100^2 - x^2}$$
$$\mu_{50.25} = \frac{100.5}{100^2 - 50.25^2} = 0.0134449$$

For UDD, we need to calculate q_{50}.

$$p_{50} = \frac{l_{51}}{l_{50}} = \frac{1-0.51^2}{1-0.50^2} = 0.986533$$
$$q_{50} = 1 - 0.986533 = 0.013467$$

so under UDD,

$$\mu^L_{50.25} = \frac{q_{50}}{1-0.25q_{50}} = \frac{0.013467}{1-0.25(0.013467)} = 0.013512.$$

The difference between $\mu_{50.25}$ and $\mu^L_{50.25}$ is $0.013445 - 0.013512 = \boxed{\mathbf{-0.000067}}$. **(B)**

7.7. $S_0(20) = 1/1.2^4$ and $S_0(21) = 1/1.21^4$, so $q_{20} = 1 - (1.2/1.21)^4 = 0.03265$. Then

$$_{0.4}q_{20.2} = \frac{0.4q_{20}}{1-0.2q_{20}} = \frac{0.4(0.03265)}{1-0.2(0.03265)} = \boxed{\mathbf{0.01315}}$$

7.8.

I. Calculate $_{1|2}q_{36}$.

$$_{1|2}q_{36} = \frac{_2d_{37}}{l_{36}} = \frac{96-87}{99} = 0.09091 \quad ✓$$

This statement does not require uniform distribution of deaths.

II. By equation (7.5),

$$\mu_{37.5} = \frac{q_{37}}{1-0.5q_{37}} = \frac{4/96}{1-2/96} = \frac{4}{94} = 0.042553 \quad ✓$$

III. Calculate $_{0.33}q_{38.5}$.

$$_{0.33}q_{38.5} = \frac{_{0.33}d_{38.5}}{l_{38.5}} = \frac{(0.33)(5)}{89.5} = 0.018436 \quad \times$$

I can't figure out what mistake you'd have to make to get 0.021. **(A)**

7.9. First calculate q_x.

$$1 - 0.75q_x = 0.25$$
$$q_x = 1$$

Then by equation (7.3), $_{0.25}q_{x+0.5} = 0.25/(1-0.5) = 0.5$, making I true.
By equation (7.2), $_{0.5}q_x = 0.5q_x = 0.5$, making II true.
By equation (7.5), $\mu_{x+0.5} = 1/(1-0.5) = 2$, making III false. **(A)**

7.10. We use equation (7.5) to back out q_x for each age.

$$\begin{aligned}
\mu_{x+0.5} &= \frac{q_x}{1-0.5q_x} \Rightarrow q_x = \frac{\mu_{x+0.5}}{1+0.5\mu_{x+0.5}}\\
q_{80} &= \frac{0.0202}{1.0101} = 0.02\\
q_{81} &= \frac{0.0408}{1.0204} = 0.04\\
q_{82} &= \frac{0.0619}{1.03095} = 0.06
\end{aligned}$$

Then by equation (7.3), ${}_{0.5}p_{80.5} = 0.98/0.99$. $p_{81} = 0.96$, and ${}_{0.5}p_{82} = 1 - 0.5(0.06) = 0.97$. Therefore

$${}_2q_{80.5} = 1 - \left(\frac{0.98}{0.99}\right)(0.96)(0.97) = \boxed{0.0782} \qquad \textbf{(A)}$$

7.11. To do this algebraically, we split the group into those who die within 0.3 years, those who die between 0.3 and 1 years, and those who survive one year. Under UDD, those who die will die at the midpoint of the interval (assuming the interval doesn't cross an integral age), so we have

Group	Survival time	Probability of group	Average survival time
I	$(0, 0.3]$	$1 - {}_{0.3}p_{x+0.7}$	0.15
II	$(0.3, 1]$	${}_{0.3}p_{x+0.7} - {}_1p_{x+0.7}$	0.65
III	$(1, \infty)$	${}_1p_{x+0.7}$	1

We calculate the required probabilities.

$$\begin{aligned}
{}_{0.3}p_{x+0.7} &= \frac{0.9}{0.93} = 0.967742\\
{}_1p_{x+0.7} &= \frac{0.9}{0.93}\bigl(1 - 0.7(0.3)\bigr) = 0.764516\\
1 - {}_{0.3}p_{x+0.7} &= 1 - 0.967742 = 0.032258\\
{}_{0.3}p_{x+0.7} - {}_1p_{x+0.7} &= 0.967742 - 0.764516 = 0.203226\\
\mathring{e}_{x+0.7:\overline{1}|} &= 0.032258(0.15) + 0.203226(0.65) + 0.764516(1) = \boxed{0.901452}
\end{aligned}$$

Alternatively, we can use trapezoids. We already know from the above solution that the heights of the first trapezoid are 1 and 0.967742, and the heights of the second trapezoid are 0.967742 and 0.764516. So the sum of the area of the two trapezoids is

$$\begin{aligned}
\mathring{e}_{x+0.7:\overline{1}|} &= (0.3)(0.5)(1 + 0.967742) + (0.7)(0.5)(0.967742 + 0.764516)\\
&= 0.295161 + 0.606290 = \boxed{0.901451}
\end{aligned}$$

7.12. For the expected value, we'll use the recursive formula. (The trapezoidal rule could also be used.)

$$\begin{aligned}
\mathring{e}_{45:\overline{2}|} &= \mathring{e}_{45:\overline{1}|} + p_{45}\,\mathring{e}_{46:\overline{1}|}\\
&= (1 - 0.005) + 0.99(1 - 0.0055)\\
&= 1.979555
\end{aligned}$$

We'll use equation (5.7)to calculate the second moment.

$$\begin{aligned}
\mathbf{E}[\min(T_{45},2)^2] &= 2\int_0^2 t\,{}_tp_x\,\mathrm{d}t\\
&= 2\left(\int_0^1 t(1-0.01t)\,\mathrm{d}t + \int_1^2 t(0.99)\big(1-0.011(t-1)\big)\,\mathrm{d}t\right)\\
&= 2\left(\frac{1}{2}-0.01\left(\frac{1}{3}\right)+0.99\left(\frac{(1.011)(2^2-1^2)}{2}-0.011\left(\frac{2^3-1^3}{3}\right)\right)\right)\\
&= 2(0.496667+1.475925) = 3.94518
\end{aligned}$$

So the variance is $3.94518 - 1.979555^2 = \boxed{\mathbf{0.02654}}$.

7.13. As discussed on page 132, by equation (7.7), the difference is

$$\frac{1}{2}\,{}_{10}q_x = \frac{1}{2}(1-0.2) = \boxed{\mathbf{0.4}}$$

7.14. Those who die in the first year survive $\frac{1}{2}$ year on the average and those who die in the first half of the second year survive 1.25 years on the average, so we have

$$\begin{aligned}
p_{60} &= 0.98\\
{}_{1.5}p_{60} &= 0.98\big(1-0.5(0.022)\big) = 0.96922\\
\mathring{e}_{60:\overline{1.5}|} &= 0.5(0.02)+1.25(0.98-0.96922)+1.5(0.96922) = \boxed{\mathbf{1.477305}} \qquad \textbf{(D)}
\end{aligned}$$

Alternatively, we use the trapezoidal method. The first trapezoid has heights 1 and $p_{60} = 0.98$ and width 1. The second trapezoid has heights $p_{60} = 0.98$ and ${}_{1.5}p_{60} = 0.96922$ and width $1/2$.

$$\begin{aligned}
\mathring{e}_{60:\overline{1.5}|} &= \frac{1}{2}(1+0.98)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(0.98+0.96922)\\
&= \boxed{\mathbf{1.477305}} \qquad \textbf{(D)}
\end{aligned}$$

7.15. $p_{70} = 1-0.040 = 0.96$, ${}_2p_{70} = (0.96)(0.956) = 0.91776$, and by linear interpolation, ${}_{1.5}p_{70} = 0.5(0.96+0.91776) = 0.93888$. Those who die in the first year survive 0.5 years on the average and those who die in the first half of the second year survive 1.25 years on the average. So

$$\mathring{e}_{70:\overline{1.5}|} = 0.5(0.04)+1.25(0.96-0.93888)+1.5(0.93888) = \boxed{\mathbf{1.45472}} \qquad \textbf{(C)}$$

Alternatively, we can use the trapezoidal method. The first year's trapezoid has heights 1 and 0.96 and width 1 and the second year's trapezoid has heights 0.96 and 0.93888 and width $1/2$, so

$$\mathring{e}_{70:\overline{1.5}|} = 0.5(1+0.96)+0.5(0.5)(0.96+0.93888) = \boxed{\mathbf{1.45472}} \qquad \textbf{(C)}$$

7.16. First we calculate ${}_tp_1$ for $t = 1, 2$.

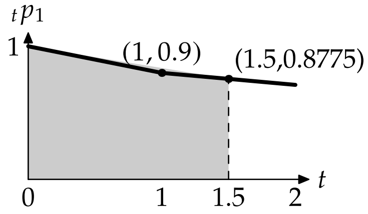
$$\begin{aligned}
p_1 &= 1-q_1 = 0.90\\
{}_2p_1 &= (1-q_1)(1-q_2) = (0.90)(0.95) = 0.855
\end{aligned}$$

By linear interpolation, ${}_{1.5}p_1 = (0.5)(0.9+0.855) = 0.8775$.

The algebraic method splits the students into three groups: first year dropouts, second year (up to time 1.5) dropouts, and survivors. In each dropout group survival on the average is to the midpoint (0.5 years for the first group, 1.25 years for the second group) and survivors survive 1.5 years. Therefore

$$\mathring{e}_{1:\overline{1.5}|} = 0.10(0.5) + (0.90 - 0.8775)(1.25) + 0.8775(1.5) = \boxed{\mathbf{1.394375}} \qquad \textbf{(D)}$$

Alternatively, we could sum the two trapezoids making up the shaded area at the right.

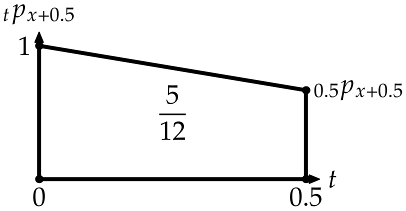


$$\begin{aligned}\mathring{e}_{1:\overline{1.5}|} &= (1)(0.5)(1 + 0.9) + (0.5)(0.5)(0.90 + 0.8775)\\ &= 0.95 + 0.444375 = \boxed{\mathbf{1.394375}} \qquad \textbf{(D)}\end{aligned}$$

7.17. Those who die survive 0.25 years on the average and survivors survive 0.5 years, so we have

$$\begin{aligned}0.25\,{}_{0.5}q_{x+0.5} + 0.5\,{}_{0.5}p_{x+0.5} &= \tfrac{5}{12}\\ 0.25\left(\frac{0.5q_x}{1-0.5q_x}\right) + 0.5\left(\frac{1-q_x}{1-0.5q_x}\right) &= \tfrac{5}{12}\\ 0.125q_x + 0.5 - 0.5q_x &= \tfrac{5}{12} - \tfrac{5}{24}q_x\\ \frac{1}{2} - \frac{5}{12} &= \left(-\frac{5}{24} + \frac{1}{2} - \frac{1}{8}\right)q_x\\ \frac{1}{12} &= \frac{q_x}{6}\\ q_x &= \boxed{\mathbf{\tfrac{1}{2}}}\end{aligned}$$

Alternatively, complete life expectancy is the area of the trapezoid shown on the right, so



$$\frac{5}{12} = 0.5(0.5)(1 + {}_{0.5}p_{x+0.5})$$

Then ${}_{0.5}p_{x+0.5} = \frac{2}{3}$, from which it follows

$$\begin{aligned}\frac{2}{3} &= \frac{1-q_x}{1-\frac{1}{2}q_x}\\ q_x &= \boxed{\mathbf{\tfrac{1}{2}}}\end{aligned}$$

7.18. Survivors live 0.4 years and those who die live 0.2 years on the average, so

$$0.396 = 0.4_{0.4}p_{55.2} + 0.2_{0.4}q_{55.2}$$

Using the formula ${}_{0.4}q_{55.2} = 0.4q_{55}/(1 - 0.2q_{55})$ (equation (7.3)), we have

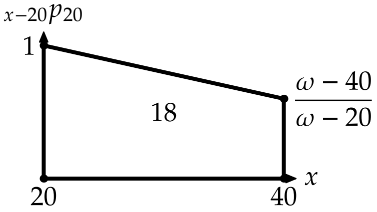
$$\begin{aligned}0.4\left(\frac{1-0.6q_{55}}{1-0.2q_{55}}\right) + 0.2\left(\frac{0.4q_{55}}{1-0.2q_{55}}\right) &= 0.396\\ 0.4 - 0.24q_{55} + 0.08q_{55} &= 0.396 - 0.0792q_{55}\\ 0.0808q_{55} &= 0.004\end{aligned}$$

$$q_{55} = \frac{0.004}{0.0808} = 0.0495$$
$$\mu_{55.2} = \frac{q_{55}}{1-0.2q_{55}} = \frac{0.0495}{1-0.2(0.0495)} = \boxed{\mathbf{0.05}}$$

7.19. Since d_x is constant for all x and deaths are uniformly distributed within each year of age, mortality is uniform globally. We back out ω using equation (5.12), $\mathring{e}_{x:\overline{n}|} = {}_np_x(n) + {}_nq_x(n/2)$:

$$\begin{aligned}
10\,{}_{20}q_{20} + 20\,{}_{20}p_{20} &= 18\\
10\left(\frac{20}{\omega-20}\right) + 20\left(\frac{\omega-40}{\omega-20}\right) &= 18\\
200 + 20\omega - 800 &= 18\omega - 360\\
2\omega &= 240\\
\omega &= 120
\end{aligned}$$

Alternatively, we can back out ω using the trapezoidal rule. Complete life expectancy is the area of the trapezoid shown to the right.



$$\begin{aligned}
\mathring{e}_{20:\overline{20}|} = 18 &= (20)(0.5)\left(1+\frac{\omega-40}{\omega-20}\right)\\
0.8 &= \frac{\omega-40}{\omega-20}\\
0.8\omega - 16 &= \omega - 40\\
0.2\omega &= 24\\
\omega &= 120
\end{aligned}$$

Once we have ω, we compute

$$_{30|10}q_{30} = \frac{10}{\omega-30} = \frac{10}{90} = \boxed{\mathbf{0.1111}} \qquad \textbf{(A)}$$

7.20. We use equation (7.5) to obtain

$$\begin{aligned}
0.5 &= \frac{q_x}{1-0.5q_x}\\
q_x &= 0.4
\end{aligned}$$

Then $\mathring{e}_{45:\overline{1}|} = 0.5\big(1+(1-0.4)\big) = \boxed{\mathbf{0.8}}$. **(E)**

7.21. According to the Illustrative Life Table, $l_{30} = 9{,}501{,}381$, so we are looking for the age x such that $l_x = 0.75(9{,}501{,}381) = 7{,}126{,}036$. This is between 67 and 68. Using linear interpolation, since $l_{67} = 7{,}201{,}635$ and $l_{68} = 7{,}018{,}432$, we have

$$x = 67 + \frac{7{,}201{,}635 - 7{,}126{,}036}{7{,}201{,}635 - 7{,}018{,}432} = 67.4127$$

This is 37.4127 years into the future. $\frac{3}{4}$ of the people collect 50,000. We need $50{,}000\left(\frac{3}{4}\right)\left(\frac{1}{1.12^{37.4127}}\right) = \boxed{\mathbf{540.32}}$ per person. **(D)**

7.22. Under constant force, ${}_sp_{x+t} = p_x^s$, so $p_x = {}_{1/4}p_{x+1/4}^4 = 0.98^4 = 0.922368$ and $q_x = 1 - 0.922368 = 0.077632$. Under uniform distribution of deaths,

$$\begin{aligned}
{}_{1/4}p_{x+1/4} &= 1 - \frac{(1/4)q_x}{1-(1/4)q_x} \\
&= 1 - \frac{(1/4)(0.077632)}{1-(1/4)(0.077632)} \\
&= 1 - 0.019792 = \boxed{\mathbf{0.980208}} \qquad \textbf{(D)}
\end{aligned}$$

7.23. Under constant force, ${}_sp_{x+t} = p_x^s$, so $p_x = 0.95^4 = 0.814506$, $q_x = 1 - 0.814506 = 0.185494$. Then under a uniform assumption,

$${}_{0.25}q_{x+0.1} = \frac{0.25q_x}{1-0.1q_x} = \frac{(0.25)(0.185494)}{1-0.1(0.185494)} = \boxed{\mathbf{0.047250}} \qquad \textbf{(B)}$$

7.24. Using constant force, μ^A is a constant equal to $-\ln p_x = -\ln 0.75 = 0.287682$. Then

$$\begin{aligned}
\mu_{x+s}^B = \frac{q_x}{1-sq_x} &= 0.287682 \\
\frac{0.25}{1-0.25s} &= 0.287682 \\
0.2877 - 0.25(0.287682)s &= 0.25 \\
s &= \frac{0.287682-0.25}{(0.25)(0.287682)} = \boxed{\mathbf{0.5239}} \qquad \textbf{(D)}
\end{aligned}$$

7.25. We integrate ${}_tp_x$ from 0 to 2. Between 0 and 1, ${}_tp_x = e^{-t\mu_x}$.

$$\int_0^1 e^{-t\mu_x}\,\mathrm{d}t = \frac{1-e^{-\mu_x}}{\mu_x} = 0.99$$

Between 1 and 2, ${}_tp_x = p_x\ {}_{t-1}p_{x+1} = 0.98e^{-(t-1)\mu_{x+1}}$.

$$\int_1^2 e^{-(t-1)\mu_{x+1}}\,\mathrm{d}t = \frac{1-e^{-\mu_{x+1}}}{\mu_{x+1}} = 0.98$$

So the answer is $0.99 + 0.98(0.98) = \boxed{\mathbf{1.9504}}$. **(C)**

7.26.

$$\begin{aligned}
\mathring{e}_{80.4:\overline{0.8}|} &= \mathring{e}_{80.4:\overline{0.6}|} + {}_{0.6}p_{80.4}\,\mathring{e}_{81:\overline{0.2}|} \\
&= \frac{\int_{0.4}^1 0.9^t\,\mathrm{d}t}{0.9^{0.4}} + 0.9^{0.6}\int_0^{0.2} 0.8^t\,\mathrm{d}t \\
&= \frac{0.9^{0.6}-1}{\ln 0.9} + \left(0.9^{0.6}\right)\frac{0.8^{0.2}-1}{\ln 0.8} \\
&= 0.581429 + (0.938740)(0.195603) = \boxed{\mathbf{0.765049}}
\end{aligned}$$

7.27. Under uniform distribution, the numbers of deaths in each half of the year are equal, so if 120 deaths occurred in the first half of $x = 96$, then 120 occurred in the second half, and $l_{97} = 480 - 120 = 360$. Then if ${}_{0.5}q_{97} = (360 - 288)/360 = 0.2$, then $q_{97} = 2\,{}_{0.5}q_{97} = 0.4$, so $p_{97} = 0.6$. Under constant force, ${}_{1/2}p_{97} = p_{97}^{0.5} = \sqrt{0.6}$. The answer is $360\sqrt{0.6} = \boxed{\mathbf{278.8548}}$. **(D)**

7.28. Let μ be the force of mortality in year 1. Then 10% survivorship means

$$\begin{aligned} e^{-\mu-3\mu} &= 0.1 \\ e^{-4\mu} &= 0.1 \end{aligned}$$

The probability of survival 21 months given survival 3 months is the probability of survival 9 months after month 3, or $e^{-(3/4)\mu}$, times the probability of survival another 9 months given survival 1 year, or $e^{-(3/4)3\mu}$, which multiplies to $e^{-3\mu} = (e^{-4\mu})^{3/4} = 0.1^{3/4} = 0.177828$, so the death probability is $1 - 0.177828 = \boxed{\mathbf{0.822172}}$. **(E)**

7.29. The exact value is:

$$F = {}_{10.5}p_0 = \exp\left(-\int_0^{10.5} \mu_x \,\mathrm{d}x\right)$$

$$\begin{aligned} \int_0^{10.5} (80 - x)^{-0.5}\mathrm{d}x &= -2(80 - x)^{0.5}\Big|_0^{10.5} \\ &= -2\left(69.5^{0.5} - 80^{0.5}\right) = 1.215212 \\ {}_{10.5}p_0 &= e^{-1.215212} = 0.296647 \end{aligned}$$

To calculate $S_0(10.5)$ with constant force interpolation between 10 and 11, we have ${}_{0.5}p_{10} = p_{10}^{0.5}$, and ${}_{10.5}p_0 = {}_{10}p_0\,{}_{0.5}p_{10}$, so

$$\begin{aligned} \int_0^{10} (80 - x)^{-0.5}\mathrm{d}x &= -2\left(70^{0.5} - 80^{0.5}\right) = 1.155343 \\ \int_{10}^{11} (80 - x)^{-0.5}\mathrm{d}x &= -2\left(69^{0.5} - 70^{0.5}\right) = 0.119953 \\ G = {}_{10.5}p_0 &= e^{-1.155343-0.5(0.119953)} = 0.296615 \end{aligned}$$

Then $F - G = 0.296647 - 0.296615 = \boxed{\mathbf{0.000032}}$. **(D)**

Quiz Solutions

7-1. Notice that $\mu_{50.4} = \frac{q_{50}}{1-0.4q_{50}}$ while ${}_{0.6}q_{50.4} = \frac{0.6q_{50}}{1-0.4q_{50}}$, so ${}_{0.6}q_{50.4} = 0.6(0.01) = \boxed{\mathbf{0.006}}$

7-2. The algebraic method goes: those who die will survive 0.3 on the average, and those who survive will survive 0.6.

$$\begin{aligned} {}_{0.6}q_{x+0.4} &= \frac{0.6(0.1)}{1 - 0.4(0.1)} = \frac{6}{96} \\ {}_{0.6}p_{x+0.4} &= 1 - \frac{6}{96} = \frac{90}{96} \\ \mathring{e}_{x+0.4:\overline{0.6}|} &= \frac{6}{96}(0.3) + \frac{90}{96}(0.6) = \frac{55.8}{96} = \boxed{\mathbf{0.58125}} \end{aligned}$$

The geometric method goes: we need the area of a trapezoid having height 1 at $x + 0.4$ and height 90/96 at $x + 1$, where 90/96 is ${}_{0.6}p_{x+0.4}$, as calculated above. The width of the trapezoid is 0.6. The answer is therefore $0.5\,(1 + 90/96)\,(0.6) = \boxed{\mathbf{0.58125}}$.

7-3. Batteries failing in month 1 survive an average of 0.5 month, those failing in month 2 survive an average of 1.5 months, and those failing in month 3 survive an average of 2.125 months (the average of 2 and 2.25). By linear interpolation, ${}_{2.25}q_0 = 0.25(0.6) + 0.75(0.2) = 0.3$. So we have

$$\begin{aligned}\mathring{e}_{0:\overline{2.25}|} &= q_0(0.5) + {}_{1|}q_0(1.5) + {}_{2|0.25}q_0(2.125) + {}_{2.25}p_0(2.25)\\ &= (0.05)(0.5) + (0.20 - 0.05)(1.5) + (0.3 - 0.2)(2.125) + 0.70(2.25) = \boxed{\mathbf{2.0375}}\end{aligned}$$

Practice Exam 1

SECTION A — Multiple-Choice

1. A life age 60 is subject to Gompertz's law with $B = 0.001$ and $c = 1.05$.

Calculate $e_{60:\overline{2}|}$ for this life.

(A) 1.923 (B) 1.928 (C) 1.933 (D) 1.938 (E) 1.943

2. For a fully discrete 20-year deferred whole life insurance of 1000 on (50), you are given:

(i) Premiums are payable for 20 years.
(ii) The net premium is 12.
(iii) Deaths are uniformly distributed between integral ages.
(iv) $i = 0.1$
(v) ${}_9V = 240$ and ${}_{9.5}V = 266.70$.

Calculate ${}_{10}V$, the net premium reserve at the end of year 10.

(A) 272.75 (B) 280.00 (C) 281.40 (D) 282.28 (E) 282.86

3. For an annual premium 2-year term insurance on (60) with benefit b payable at the end of the year of death, you are given

(i)

t	p_{60+t-1}
1	0.98
2	0.96

(ii) The annual net premium is 25.41.
(iii) $i = 0.05$.

Determine the revised annual net premium if an interest rate of $i = 0.04$ is used.

(A) 25.59 (B) 25.65 (C) 25.70 (D) 25.75 (E) 25.81

4. In a double-decrement model, with decrements (1) and (2), you are given, for all $t > 0$:

(i) ${}_tp_x^{\prime(1)} = 10/(10+t)$
(ii) ${}_tp_x^{\prime(2)} = \big(10/(10+t)\big)^3$

Determine $q_x^{(1)}$.

(A) 0.068 (B) 0.074 (C) 0.079 (D) 0.083 (E) 0.091

Exam questions continue on the next page . . .

5. A Type A universal life policy with death benefit 10,000 is sold to a person age 75. You are given the following information concerning charges and credits:

(i) 20% of premium is charged at the beginning of the first year.
(ii) The COI charge in the first year is based on $q_{75} = 0.02$.
(iii) Interest is credited on the account value at 4.5% effective.
(iv) A different interest rate is used to discount the COI.
(v) The account value is updated annually.

The policyholder contributes 1000 initially. At the end of the first year, the account value is 644.30.

Determine the interest rate used to discount the COI.

(A) 0.020 (B) 0.022 (C) 0.024 (D) 0.026 (E) 0.028

6. In a three-state Markov chain, you are given the following forces of transition:

$$\mu_t^{01} = 0.05 \qquad \mu_t^{10} = 0.04 \qquad \mu_t^{02} = 0.03 \qquad \mu_t^{12} = 0.10$$

All other forces of transition are 0.

Calculate the probability of an entity in state 0 at time 0 transitioning to state 1 before time 5 and staying there until time 5, then transitioning to state 0 before time 10 and staying there until time 10.

(A) 0.017 (B) 0.018 (C) 0.019 (D) 0.020 (E) 0.021

7. For a temporary life annuity-due of 1 per year on (30), you are given:

(i) The annuity makes 20 certain payments.
(ii) The annuity will not make more than 40 payments.
(iii) Mortality follows the Illustrative Life Table.
(iv) $i = 0.06$

Determine the expected present value of the annuity.

(A) 14.79 (B) 15.22 (C) 15.47 (D) 15.63 (E) 16.06

Exam questions continue on the next page . . .

8. For a fully discrete whole life insurance on (35) with face amount 100,000, you are given the following assumptions and experience for the fifth year:

	Assumptions	Actual
q_{39}	0.005	0.006
Surrender probability	0.05	0.06
Annual expenses	20	30
Settlement expenses—death	100	80
Settlement expenses—surrender	50	40
i	0.05	0.045

You are also given:

(i) The gross premium is 1725.
(ii) Reserves are gross premium reserves.
(iii) The gross premium reserve at the end of year 4 is 6000.
(iv) The cash surrender value for the fifth year is 6830.
(v) The surrender probability is based on the multiple-decrement table.

The fifth year gain is analyzed in the order of interest, surrender, death, expense.

Determine the fifth year surrender gain.

(A) −7.9 (B) −7.7 (C) −7.5 (D) 7.7 (E) 7.9

9. For a defined benefit pension plan, you are given

(i) Accrual rate is 1.6%
(ii) The pension benefit is a monthly annuity-due payable starting at age 65, based on final salary.
(iii) No benefits are payable for death in service.
(iv) There are no exits other than death before retirement.
(v) Salaries increase 3% per year.
(vi) $i = 0.04$

An employee enters the plan at age 32. At age 45, the accrued liability for the pension, using the projected unit credit method, is 324,645.

Calculate the normal contribution for this employee for the year beginning at age 45.

(A) 24,000 (B) 25,000 (C) 26,000 (D) 27,000 (E) 28,000

10. For an insurance with face amount 100,000, you are given:

(i)

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = 100$$

(ii) $P = 1380$
(iii) $\delta = 0.05$
(iv) $\mu_{x+t} = 0.03$

Determine ${}_tV$.

(A) 21,000 (B) 21,500 (C) 22,000 (D) 22,500 (E) 23,000

Exam questions continue on the next page . . .

11. A life age 90 is subject to mortality following Makeham's law with $A = 0.0005$, $B = 0.0008$, and $c = 1.07$.

Curtate life expectancy for this life is 6.647 years.

Using Woolhouse's formula with three terms, compute complete life expectancy for this life.

(A) 7.118 (B) 7.133 (C) 7.147 (D) 7.161 (E) 7.176

12. For a fully continuous whole life insurance of 1000 on (x):

(i) The gross premium is paid at an annual rate of 25.
(ii) The variance of future loss is 2,000,000.
(iii) $\delta = 0.06$

Employees are able to obtain this insurance for a 20% discount.

Determine the variance of future loss for insurance sold to employees.

(A) 1,281,533 (B) 1,295,044 (C) 1,771,626 (D) 1,777,778 (E) 1,825,013

13. You are given the following profit test for a 10-year term insurance of 100,000 on (x):

t	${}_{t-1}V$	P	E_t	I_t	bq_{x+t-1}	$p_{x+t-1}\,{}_tV$
0			−350			
1	0	1000	0	60.0	500	447.75
2	450	1000	20	85.8	600	795.20
3	800	1000	20	106.8	700	1092.30
4	1100	1000	20	124.8	800	1289.60
5	1300	1000	20	136.8	900	1412.18
6	1425	1000	20	144.3	1000	1435.50
7	1450	1000	20	145.8	1100	1285.70
8	1300	1000	20	136.8	1200	1037.40
9	1050	1000	20	121.8	1300	641.55
10	650	1000	20	97.8	1400	0.00

Which of the following statements is true?

I. The interest rate used in the calculation is $i = 0.06$.
II. At time 5, the reserve per survivor is 1425.
III. The profit signature component for year 3 is 92.81

(A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

Exam questions continue on the next page . . .

14. Your company sells whole life insurance policies. At a meeting with the Enterprise Risk Management Committee, it was agreed that you would limit the face amount of the policies sold so that the probability that the present value of the benefit at issue is greater than 1,000,000 is never more than 0.05.

You are given:

(i) The insurance policies pay a benefit equal to the face amount b at the moment of death.
(ii) The force of mortality is $\mu_x = 0.001(1.05^x)$, $x > 0$
(iii) $\delta = 0.06$

Determine the largest face amount b for a policy sold to a purchaser who is age 45.

(A) 1,350,000 (B) 1,400,000 (C) 1,450,000 (D) 1,500,000 (E) 1,550,000

15. A Type A universal life policy with face amount 20,000 is issued to (50). The policy has a no-lapse guarantee, and remains in force as long as the policyholder pays a premium of 500 at the beginning of each year.

At time 10, the account value is 0, and the no-lapse guarantee is effective. The following assumptions are used for calculating the reserve:

(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.06$.
(iii) Expenses are 3% of premium plus 10, paid at the beginning of each year.
(iv) Death benefits are paid at the end of the year.

Calculate the gross premium reserve.

(A) 1992 (B) 2020 (C) 2042 (D) 2065 (E) 2089

16. For two lives (50) and (60) with independent future lifetimes:

(i) $\mu_{50+t} = 0.002t, \quad t > 0$
(ii) $\mu_{60+t} = 0.003t, \quad t > 0$

Calculate ${}_{20}q_{\overset{1}{50}:60} - {}_{20}q_{50:\overset{2}{60}}$.

(A) 0.17 (B) 0.18 (C) 0.30 (D) 0.31 (E) 0.37

17. You are given that $\mu_x = 0.002x + 0.005$.

Calculate ${}_{5|}q_{20}$.

(A) 0.015 (B) 0.026 (C) 0.034 (D) 0.042 (E) 0.050

Exam questions continue on the next page . . .

18. For a 30-pay whole life insurance policy of 100,000 on (45), you are given:

(i) Benefits are payable at the end of the year of death.
(ii) Premiums and expenses are payable at the beginning of the year.
(iii) $\ddot{a}_{45} = 14.1121$
(iv) $\ddot{a}_{45:\overline{30}|} = 13.3722$
(v) $i = 0.06$
(vi) Expenses are:

	Per Premium	Per Policy
First Year	40%	200
Renewal Years	10%	r
Settlement		100

(vii) The gross premium determined by the equivalence principle is 1777.98.

Determine r.

(A) 37 (B) 38 (C) 39 (D) 40 (E) 41

19. For a special fully discrete whole life insurance on (40), you are given:

(i) The annual net premium in the first 20 years is $1000P_{40}$.
(ii) The annual net premium changes at age 60.
(iii) The death benefit is 1000 in the first 20 years, after which it is 2000.
(iv) Mortality follows the Illustrative Life Table.
(v) $i = 0.06$

Determine ${}_{21}V$, the net premium reserve for the policy at the end of 21 years.

(A) 282 (B) 286 (C) 292 (D) 296 (E) 300

20. You are given the following yield curve:

$$y_t = \begin{cases} 0.01 + 0.004t & 0 < t \leq 5 \\ 0.02 + 0.002t & 5 \leq t \leq 20 \\ 0.06 & t \geq 20 \end{cases}$$

Calculate the 2-year forward rate on a 10-year zero-coupon bond.

(A) 0.040 (B) 0.044 (C) 0.047 (D) 0.049 (E) 0.052

Exam questions continue on the next page . . .

SECTION B — Written-Answer

1. (11 points) A special 5-year term insurance on (55) pays 1000 plus the net premium reserve at the end of the year of death. A single premium is paid at inception. You are given:

(i) Mortality follows the Illustrative Life Table.
(ii) $i = 0.06$

(a) (2 points) Calculate the net single premium for this policy.

(b) (3 points) Using the recursive formula for reserves, calculate net premium reserves for the policy at times 1, 2, 3, and 4.

(c) (2 points) Suppose the policy, in addition to paying death benefits, pays the single premium at the end of 5 years to those who survive.
Calculate the revised single premium.

(d) (2 points) Calculate the net single premium for an otherwise similar policy that pays 1000, but not the net premium reserve, at the end of the year of death.

(e) (2 points) Calculate the net single premium for an otherwise similar policy that pays 1000 plus the net single premium, but not the net premium reserve, at the end of the year of death.

2. (7 points) For a Type B universal life policy on (50) with face amount 100,000:

(i) The following charges and credits are made to the policy:
  1. Expense charge is 500 per year.
  2. COI is based on $q_{50+t} = 0.01 + 0.001t$.
  3. Interest is credited at $i = 0.04$.
  4. Surrender charge in year t is $1200 - 200t$ for $t = 1, 2, 3, 4, 5$.
  5. The account value is updated annually.

(ii) The following assumptions are made:
  1. Expenses are 400 per year.
  2. Mortality is $q'^{(\text{death})}_{50+t} = 0.009 + 0.001t$.
  3. Surrender rate is $q'^{(\text{surrender})}_{50+t} = 0.06$ for all t, with all surrenders occurring at the end of the year.
  4. Interest is earned at $i = 0.04$.
  5. There are no settlement expenses.

(iii) The account value at time 4 is 10,000.
(iv) The reserve equals the account value.
(v) The policyholder does not pay a premium in the fifth year.

(a) (2 points) Calculate the account value at the end of year 5.

(b) (3 points) Calculate the expected profit in year 5 per policy issued.

(c) (2 points) The corridor factor in the fifth year is 1.57.
Determine the largest amount that the policyholder can pay at the beginning of year 5 without forcing the face amount to increase due to the corridor factor.

Exam questions continue on the next page . . .

3. (9 points) A one-year term life insurance on (x) pays 2000 at the moment of decrement 1 and 1000 at the moment of decrement 2. You are given

(i) $q_x'^{(1)} = 0.1$
(ii) $q_x'^{(2)} = 0.3$
(iii) $\delta = 0.04$

(a) (3 points) The decrements are uniform in the multiple decrement table.
Calculate the EPV of the insurance.

(b) (3 points) The decrements are uniform in the associated single decrement tables.
Calculate the EPV of the insurance.

(c) (3 points) The forces of decrement are constant.
Calculate the EPV of the insurance.

4. (8 points) A continuous whole life annuity on (60) pays 100 per year.

You are given:

(i) Mortality follows $l_x = 1000(100 - x)$, $0 \le x \le 100$.
(ii) $\delta = 0.05$.

(a) (2 points) Calculate the probability that the present value of payments on the annuity is greater than its net single premium.

Use the following information for (b) and (c):

In addition to the annuity payments, a death benefit of 1000 is paid at the moment of death if death occurs within the first ten years.

(b) (4 points) Calculate the probability that the present value of payments on the annuity (including the death benefit) is greater than its net single premium.

(c) (2 points) Calculate the minimum value of the present value of payments.

Exam questions continue on the next page . . .

5. (10 points) A special whole life insurance on (35) pays a benefit at the moment of death. You are given:

(i) The benefit for death in year k is $9000 + 1000k$, but never more than 20,000.
(ii) Mortality follows the Illustrative Life Table.
(iii) $i = 0.06$.
(iv) $1000(IA)^{1}_{35:\overline{10}|} = 107.98$
(v) Premiums are payable monthly.

(a) (2 points) Calculate the net single premium for the policy assuming uniform distribution of deaths between integral ages.

(b) (2 points) Calculate the net single premium for a whole life annuity-due annuity on (35) of 1 per month using Woolhouse's formula and approximating $\mu_x = -0.5(\ln p_{x-1} + \ln p_x)$.

(c) (1 point) Calculate the net premium payable monthly, using the assumptions and methods of parts (a) and (b).

(d) (3 points) Calculate the net premium reserve at time 10, using the same method as was used to calculate the net premium.

Suppose that instead of the benefit pattern of (i), the death benefit of the insurance is $11{,}000 - 1000k$, but never less than 1000.

(e) (2 points) Calculate the net single premium for the insurance, assuming uniform distribution of deaths between integral ages.

6. (11 points) The ZYX Company offers a defined benefit pension plan with the following provisions:

- At retirement at age 65, the plan pays a monthly whole life annuity-due providing annual income that accrues at the rate of 1.5% of final salary up to 100,000 and 2% of the excess of final salary over 100,000 for each year of service.
- There is no early retirement.
- There are no other benefits.

The following assumptions are made:

(i) No employees exit the plan before retirement except by death.
(ii) Retirement occurs at the beginning of each year.
(iii) Pre-retirement mortality follows the Illustrative Life Table.
(iv) Salaries increase 3% each year.
(v) $i = 0.06$.
(vi) $\ddot{a}_{65}^{(12)} = 11$.

The ZYX Company has the following 3 employees on January 1, 2015:

Name	Exact Age	Years of Service	Salary in Previous Year
Cramer	55	20	120,000
Liu	35	5	50,000
Smith	50	10	100,000

(a) (3 points) Show that the actuarial liability using TUC is 267,000 to the nearest 1000. You should answer to the nearest 10.

(b) (3 points) Calculate the normal contribution for the year using TUC.

(c) (1 point) Calculate the replacement ratio for Cramer if he retires at age 65 and the salary increases follow assumptions.

(d) (2 points) Fifteen years later, Smith retires. Smith's salary increases have followed assumptions. Smith would prefer an annual whole life annuity-due.

Calculate the annual payment that is equivalent to the pension plan's monthly benefit using Woolhouse's formula to two terms.

(e) (2 points) On January 2, 2015, a pension consultant suggests that $q_{39} = 0.00244$ is a better estimate of mortality than the rate in the Illustrative Life Table. No other mortality rate changes are suggested.

Recalculate the actuarial liability under TUC as of January 1, 2015 using this new assumption.

Solutions to the above questions begin on page 1577.

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	E	6	A	11	A	16	B
2	D	7	C	12	C	17	D
3	C	8	E	13	A	18	D
4	C	9	B	14	A	19	B
5	A	10	B	15	E	20	D

Practice Exam 1

SECTION A — Multiple-Choice

1. [**Section 5.2**] By formula (4.2),

$$p_{60} = \exp\left(-0.001(1.05^{60})\left(\frac{0.05}{\ln 1.05}\right)\right) = 0.981040$$
$${}_2p_{60} = \exp\left(-0.001(1.05^{60})\left(\frac{1.05^2-1}{\ln 1.05}\right)\right) = 0.961518$$

Then $e_{60:\overline{2}|} = 0.981040 + 0.961518 = \boxed{\mathbf{1.9426}}$. **(E)**

2. [**Section 41.2**] We need to back out q_{59}. We use reserve recursion. Since the insurance is deferred, $1000q_{59}$ is not subtracted from the left side.

$$({}_9V + P)(1.1^{0.5}) = {}_{9.5}V(1-0.5q_{59})$$
$$252(1.1^{0.5}) = 266.70 - 133.35q_{59}$$
$$q_{59} = \frac{2.40017}{133.35} = 0.018$$

Then the net premium reserve at time 10 is, by recursion from time 9,

$$\frac{252(1.1)}{1-0.018} = \boxed{\mathbf{282.28}} \qquad \textbf{(D)}$$

3. [**Lesson 24**] The revised premium for the entire policy is 25.41 times the ratio of the revised premium per unit at 4% to the original premium per unit at 5%.

We calculate the original net premium per unit, $P^{1}_{60:\overline{2}|}$.

$$\ddot{a}_{60:\overline{2}|} = 1 + \frac{0.98}{1.05} = 1.93333$$
$$A^{1}_{60:\overline{2}|} = \frac{0.02}{1.05} + \frac{(0.98)(0.04)}{1.05^2} = 0.054603$$
$$P^{1}_{60:\overline{2}|} = \frac{A^{1}_{60:\overline{2}|}}{\ddot{a}_{60:\overline{2}|}} = \frac{0.054603}{1.93333} = 0.028243$$

Now we recalculate at 4%. Call the revised premium $P'^{\,1}_{60:\overline{2|}}$.

$$\begin{aligned}
\ddot{a}_{60:\overline{2|}} &= 1 + \frac{0.98}{1.04} = 1.94231\\
A^{\,1}_{60:\overline{2|}} &= \frac{0.02}{1.04} + \frac{(0.98)(0.04)}{1.04^2} = 0.055473\\
P'^{\,1}_{60:\overline{2|}} &= \frac{0.055473}{1.94231} = 0.028561
\end{aligned}$$

So the revised premium for benefit b is 25.41(0.028561/0.028243) = $\boxed{\mathbf{25.696}}$. **(C)**

4. **[Lesson 48]**

$$\begin{aligned}
{}_tp_x^{(\tau)} &= \left(\frac{10}{10+t}\right)\left(\frac{10}{10+t}\right)^3 = \left(\frac{10}{10+t}\right)^4\\
\mu_{x+t}^{(1)} &= -\frac{\mathrm{d}\ln {}_tp_x'^{(1)}}{\mathrm{d}t}\\
&= -\frac{\mathrm{d}\big(\ln 10 - \ln(10+t)\big)}{\mathrm{d}t}\\
&= \frac{1}{10+t}\\
q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\,\mu_{x+t}^{(1)}\,\mathrm{d}t\\
&= \int_0^1 \left(\frac{10}{10+t}\right)^4\left(\frac{1}{10+t}\right)\mathrm{d}t\\
&= \int_0^1 \frac{10^4\mathrm{d}t}{(10+t)^5}\\
&= -\left(\frac{10^4}{4}\right)\left(\frac{1}{(10+t)^4}\right)\bigg|_0^1\\
&= \left(\frac{10^4}{4}\right)\left(\frac{1}{10^4} - \frac{1}{11^4}\right)\\
&= \boxed{\mathbf{0.079247}} \qquad \textbf{(C)}
\end{aligned}$$

5. **[Section 67.1]** Use the formula relating account values. Let $v_q = 1/(1+i_q)$ be the discount factor for COI.

$$\begin{aligned}
\mathrm{AV}_1 &= \frac{(P - E - v_q q_{75} FA)(1+i)}{1 - v_q(1+i)q_{75}}\\
644.30 &= \frac{(1000 - 200 - 200v_q)(1.045)}{1 - 1.045v_q(0.02)}\\
644.30 - 13.4659v_q &= 836 - 209v_q\\
195.5341v_q &= 191.7\\
v_q &= 0.9803915\\
i_q &= \frac{1}{0.9803915} - 1 = \boxed{\mathbf{0.02}} \qquad \textbf{(A)}
\end{aligned}$$

6. **[Section 44.1]** Let ${}_5p_0^{\overline{01}}$ be the probability that an entity in state 0 at time 0 transitions to state 1 before time 5 and stays there until time 5, and let ${}_5p_5^{\overline{10}}$ be the probability that an entity in state 1 at time 5 transitions to state 0 before time 10 and stays there until time 10. We'll use formula (44.9) for both transitions. Notice that the formula is the same with 0 and 1 switched, except that ${}_5p_0^{\overline{01}}$ uses $\mu^{01} = 0.05$ and ${}_5p_5^{\overline{10}}$ uses $\mu^{10} = 0.04$ outside the parentheses.

$$\frac{e^{-\mu^{0\bullet}t}}{\mu^{1\bullet} - \mu^{0\bullet}} + \frac{e^{-\mu^{1\bullet}t}}{\mu^{0\bullet} - \mu^{1\bullet}} = \frac{e^{-0.08(5)}}{0.14 - 0.08} + \frac{e^{-0.14(5)}}{0.08 - 0.14} = 2.89558$$

$${}_5p_0^{\overline{01}} = 0.05(2.89558) = 0.14478$$

$${}_5p_5^{\overline{10}} = 0.04(2.89558) = 0.11582$$

The answer is (0.14478)(0.11582) = $\boxed{\mathbf{0.01677}}$. **(A)**

7. **[Lesson 17]** This annuity is the sum of a 20-year certain annuity-due and a 20-year deferred 20-year temporary life annuity due.

$$\begin{aligned}
\ddot{a}_{\overline{20}|} &= \frac{1 - (1/1.06)^{20}}{1 - 1/1.06} = 12.15812\\
{}_{20|}\ddot{a}_{30:\overline{20}|} &= {}_{20|}\ddot{a}_{30} - {}_{40|}\ddot{a}_{30}\\
&= {}_{20}E_{30}\,\ddot{a}_{50} - {}_{40}E_{30}\,\ddot{a}_{70}\\
&= {}_{20}E_{30}\,\ddot{a}_{50} - {}_{20}E_{30}\,{}_{20}E_{50}\,\ddot{a}_{70}\\
&= (0.29374)(13.2668) - (0.29374)(0.23047)(8.5693)\\
&= 3.89699 - (0.067699)(8.5693)\\
&= 3.89699 - 0.58013 = 3.31686
\end{aligned}$$

The expected present value of the annuity is 12.15812 + 3.31686 = $\boxed{\mathbf{15.4750}}$. **(C)**

8. **[Lesson 68]** Surrender gain per surrender is the ending reserve (which is released into profit) minus the benefit paid and minus expenses. The ending gross premium reserve is

$${}_5V = \frac{(6000 + 1725 - 20)(1.05) - (100{,}000 + 100)(0.005) - (6830 + 50)(0.05)}{1 - 0.05 - 0.005} = 7667.46$$

Using assumed expenses, the surrender gain per surrender is 7667.46 − (6830 + 50) = 787.46. The gain is (0.06 − 0.05)(787.46) = $\boxed{\mathbf{7.8746}}$. **(E)**

9. **[Section 61.4]** Using PUC, if there are no exit benefits and accruals are the same percentage each year, the normal contribution is the initial accrued liability divided by the number of years of service, or 324,645/13 = $\boxed{\mathbf{24{,}973}}$. **(B)**

10. **[Section 41.3]**

$$100 = (0.05 + 0.03)_tV + 1380 - 100{,}000(0.03) = 0.08_tV - 1620$$

$${}_tV = \frac{1720}{0.08} = \boxed{\mathbf{21{,}500}} \qquad \textbf{(B)}$$

11. [**Section 22.2**] By equation (22.10),

$$\mathring{e}_x = e_x + \tfrac{1}{2} - \tfrac{1}{12}\mu_x$$

Force of mortality for (90) is $\mu_{90} = 0.0005 + 0.0008(1.07^{90}) = 0.353382$. Thus

$$\mathring{e}_{90} = 6.647 + 0.5 - \tfrac{1}{12}(0.353382) = \boxed{\mathbf{7.118}} \qquad \textbf{(A)}$$

12. [**Lesson 30**] The variance of future loss for a gross premium of 25 is

$$\begin{aligned} 2{,}000{,}000 &= \operatorname{Var}\left(v^{T_x}\right)\left(1000 + \frac{25}{0.06}\right)^2 \\ &= \operatorname{Var}\left(v^{T_x}\right)(2{,}006{,}944) \end{aligned}$$

If we replace 25 with 20 (for a 20% discount) in the above formula, it becomes

$$\begin{aligned} \operatorname{Var}\left({}_0L\right) &= \operatorname{Var}\left(v^{T_x}\right)\left(1000 + \frac{20}{0.06}\right)^2 \\ &= \operatorname{Var}\left(v^{T_x}\right)(1{,}777{,}778) \end{aligned}$$

We see that this is 1,777,778/2,006,944 times the given variance, so the final answer is

$$\operatorname{Var}({}_0L) = \frac{1{,}777{,}778}{2{,}006{,}944}(2{,}000{,}000) = \boxed{\mathbf{1{,}771{,}626}} \qquad \textbf{(C)}$$

13. [**Lesson 65**]

I From the row for year 1, with 0 reserves and expenses, we see that $I_t/P_t = 0.06$, so the interest rate is 0.06. ✓

II Looking at the line for $t = 6$, we see that the reserve per survivor to time $t - 1 = 5$ is 1425. ✓

III First, the profit in year 3 is $800 + 1000 - 20 + 106.8 - 700 - 1092.3 = 94.50$. We deduce survivorship from the bq_{x+t-1} column, and we see that the mortality rates in the first two years are 0.005 and 0.006, so the profit signature component of year‘3 is (0.995)(0.994)(94.50) = 93.46. ✗.

(A)

14. [**Lesson 13**] The present value of the benefit decreases with increasing survival time, so the 95th percentile of the present value of the insurance corresponds to the 5th percentile of survival time. The survival probability is

$$\begin{aligned} {}_tp_{45} &= \exp\left(-\int_0^t 0.001(1.05^{45+u})\,\mathrm{d}u\right) \\ -\ln {}_tp_{45} &= \left.\frac{0.001(1.05^{45+u})}{\ln 1.05}\right|_0^t \\ &= \frac{0.001(1.05^{45+t} - 1.05^{45})}{\ln 1.05} \end{aligned}$$

Setting ${}_tp_{45} = 0.95$,

$$\begin{aligned}\frac{0.001(1.05^{45+t} - 1.05^{45})}{\ln 1.05} &= -\ln 0.95\\ 1.05^{45+t} &= (-1000\ln 0.95)(\ln 1.05) + 1.05^{45} = 11.48762\\ 1.05^t &= \frac{11.48762}{1.05^{45}} = 1.27853\\ t &= \frac{\ln 1.27853}{\ln 1.05} = 5.0361\end{aligned}$$

The value of Z if death occurs at $t = 5.0361$ is $be^{-5.0361(0.06)}$, so the largest face amount is $1{,}000{,}000e^{5.0361(0.06)} = \boxed{\mathbf{1{,}352{,}786}}$. **(A)**

15. **[Section 67.1]** The expected present value of future benefits and expenses is

$$20{,}000A_{60} + \bigl(10 + 0.03(500)\bigr)\ddot{a}_{60} = 20(369.13) + 25(11.1454) = 7661.24$$

The expected present value of future premiums is $500\ddot{a}_{60} = 500(11.1454) = 5572.70$. The gross premium reserve is $7661.24 - 5572.70 = \boxed{\mathbf{2088.54}}$. **(E)**

16. **[Lesson 56]** ${}_{20}q^{1}_{50:60} - {}_{20}q^{2}_{50:60} = {}_{20}q_{50}\,{}_{20}p_{60}$, and

$$\begin{aligned}{}_{20}q_{50} &= 1 - \exp\left(-\int_0^{20} 0.002t\,\mathrm{d}t\right)\\ &= 1 - e^{-0.001(20)^2} = 1 - 0.670320 = 0.329680\\ {}_{20}p_{60} &= \exp\left(-\int_0^{20} 0.003t\,\mathrm{d}t\right)\\ &= e^{-0.0015(20)^2} = 0.548812\\ {}_{20}q_{50}\,{}_{20}p_{60} &= (0.329680)(0.548812) = \boxed{\mathbf{0.180932}} \qquad \textbf{(B)}\end{aligned}$$

17. **[Lesson 3]** ${}_{5|}q_{20} = \bigl(S_0(25) - S_0(26)\bigr)/S_0(20)$, so we will calculate these three values of $S_0(x)$. (Equivalently, one could calculate ${}_5p_{20}$ and ${}_6p_{20}$ and take the difference.) The integral of μ_x is

$$\int_0^x \mu_u\,\mathrm{d}u = \left(\frac{0.002u^2}{2} + 0.005u\right)\Bigg|_0^x = 0.001x^2 + 0.005x$$

so

$$\begin{aligned}S_0(20) &= \exp\Bigl(-\bigl(0.001(20^2) + 0.005(20)\bigr)\Bigr) = \exp(-0.5) = 0.606531\\ S_0(25) &= \exp\Bigl(-\bigl(0.001(25^2) + 0.005(25)\bigr)\Bigr) = \exp(-0.75) = 0.472367\\ S_0(26) &= \exp\Bigl(-\bigl(0.001(26^2) + 0.005(26)\bigr)\Bigr) = \exp(-0.806) = 0.446641\end{aligned}$$

and the answer is

$${}_{5|}q_{20} = \frac{0.472367 - 0.446641}{0.606531} = \boxed{\mathbf{0.042415}} \qquad \textbf{(D)}$$

18. **[Lesson 28]** By the equivalence principle,

$$
\begin{aligned}
G(0.9\ddot{a}_{45:\overline{30}|} - 0.3) &= 100{,}100A_{45} + ra_{45} + 200 && (*)\\
1000A_{45} &= 1000\,(1 - d\ddot{a}_{45}) = 1000\left(1 - \frac{0.06}{1.06}(14.1121)\right) = 201.2\\
a_{45} &= 14.1121 - 1 = 13.1121\\
0.9\ddot{a}_{45:\overline{30}|} - 0.3 &= 0.9(13.3722) - 0.3 = 11.7350
\end{aligned}
$$

Substituting into (*),

$$
\begin{aligned}
1777.98(11.7350) &= 100.1(201.2) + 13.1121r + 200\\
r &= \frac{1777.98(11.7350) - 100.1(201.2) - 200}{13.1121} = \boxed{\mathbf{40}} \qquad \textbf{(D)}
\end{aligned}
$$

19. **[Lessons 36 and 39]** Because premiums and benefits are the same as for an insurance on (40) through year 20, ${}_{20}V$ must be the same as for a standard 1000 whole life insurance on (40), or

$$
{}_{20}V_{40} = 1 - \frac{\ddot{a}_{60}}{\ddot{a}_{40}} = 1 - \frac{11.1454}{14.8166} = 0.247776
$$

Then by the equivalence principle, this reserve plus expected future net premiums must equal expected future benefits. If we let P be the premium after age 60:

$$
\begin{aligned}
2000A_{60} &= 247.776 + P\ddot{a}_{60}\\
2000(0.36913) &= 247.776 + P(11.1454)\\
P &= \frac{2000(0.36913) - 247.776}{11.1454} = 44.0077
\end{aligned}
$$

Now we roll the reserve forward one year.

$$
\begin{aligned}
{}_{21}V &= \frac{({}_{20}V + P)(1+i) - 2000q_{60}}{1 - q_{60}}\\
&= \frac{(247.776 + 44.0077)(1.06) - 2000(0.01376)}{1 - 0.01376}\\
&= \boxed{\mathbf{285.70}} \qquad \textbf{(B)}
\end{aligned}
$$

20. **[Lesson 62]**

$$
\begin{aligned}
y_2 &= 0.018\\
y_{12} &= 0.044\\
(1 + f(2,12))^{10} &= \frac{1.044^{12}}{1.018^2} = 1.617746\\
f(2,12) &= \sqrt[10]{1.617446} - 1 = \boxed{\mathbf{0.0493}} \qquad \textbf{(D)}
\end{aligned}
$$

SECTION B — Written-Answer

1. [Section 39.2]

(a) The reserve at time 5 is 0, so the single premium P is determined from

$$0 = P(1+i)^5 - 1000\sum_{k=1}^{5} q_{55+k-1}(1+i)^{5-k}$$

or

$$\begin{aligned}
P &= 1000\sum_{k=1}^{5} q_{55+k-1}v^k \\
&= 1000\left(\frac{0.00896}{1.06} + \frac{0.00975}{1.06^2} + \frac{0.01062}{1.06^3} + \frac{0.01158}{1.06^4} + \frac{0.01262}{1.06^5}\right) \\
&= \boxed{\mathbf{44.6499}}
\end{aligned}$$

(b) Because the net premium reserve is paid on death, the recursion does not divide by p_x.

$$\begin{aligned}
44.6499(1.06) - 8.96 &= 38.3689 \\
38.3689(1.06) - 9.75 &= 30.9210 \\
30.9210(1.06) - 10.62 &= 22.1563 \\
22.1563(1.06) - 11.58 &= 11.9057
\end{aligned}$$

Although not required, you could check the calculation by doing one more recursion: $11.9057(1.06) - 12.62 = 0$.

(c) The reserve at time 5 is P, so the single premium P is determined from

$$P = P(1+i)^5 - 1000\sum_{k=1}^{5} q_{55+k-1}(1+i)^{5-k}$$

or

$$P(1-v^5) = 1000\sum_{k=1}^{5} q_{55+k-1}v^k$$

We divide the answer to part (a) by $1 - v^5$:

$$44.6499/(1 - 1/1.06^5) = \boxed{\mathbf{176.6621}}$$

(d)

$$1000A^{\,1}_{55:\overline{5|}} = 1000(A_{55} - {}_5E_{55}\,A_{60}) = 305.14 - (0.70810)(369.13) = \boxed{\mathbf{43.7590}}$$

(e)

$$\begin{aligned}
P &= (1000+P)A^{\,1}_{55:\overline{5|}} \\
P &= \frac{43.7590}{1 - 0.0437590} = \boxed{\mathbf{45.7615}}
\end{aligned}$$

2. [Section 67.2]

(a)

$$(10{,}000-500)(1.04)-0.014(100{,}000)=\boxed{\mathbf{8480}}$$

(b) Calculate profit per policy in force at the beginning of year 5. Expected death benefit is $0.013(100{,}000+8{,}480)=1410.24$. Expected surrender benefit is $(0.987)(0.06)(8{,}480-200)=490.34$. Expected ending account value is $(0.987)(0.94)(8{,}480)=7{,}867.57$. So profit per policy in force at the beginning of year 5 is

$$(10{,}000-400)(1.04)-1410.24-490.34-7867.57=215.84$$

Persistency to the beginning of year 4 is persistency from surrenders times persistency from deaths, or $(0.94^4)(0.991)(0.99)(0.989)(0.988)=0.748468$. So profit per policy issued is $215.84(0.748468)=\boxed{\mathbf{161.55}}$.

(c) The death benefit must not be more than 1.57 times the account value:

$$\begin{aligned}1.57\,\mathrm{AV}_5 &\le 100{,}000+\mathrm{AV}_5\\ \mathrm{AV}_5 &\le \frac{100{,}000}{0.57}=175{,}439\end{aligned}$$

AV_5 in terms of the premium P is

$$(10{,}000+P-500)(1.04)-0.014(100{,}000)=\mathrm{AV}_5=175{,}439$$

It follows that the maximum P is

$$P=\frac{175{,}439+1{,}400}{1.04}-9500=\boxed{\mathbf{160{,}537}}$$

3. [Lessons 49 and 51]

(a)

$$\begin{aligned}p_x^{(\tau)}&=(0.9)(0.7)=0.63\\ q_x^{(1)}&=(0.37)\left(\frac{\ln 0.9}{\ln 0.63}\right)=0.084373\\ q_x^{(2)}&=(0.37)\left(\frac{\ln 0.7}{\ln 0.63}\right)=0.285627\end{aligned}$$

Since the decrements are uniform in the multiple decrement table, ${}_sp_x^{(\tau)}\,\mu_{x+s}^{(j)}$ is constant and equal to $q_x^{(j)}$. The EPV of the insurance is

$$\int_0^1 v^s\,{}_sp_x^{(\tau)}(2000\mu_{x+s}^{(1)}+1000\mu_{x+s}^{(2)})\mathrm{d}s=\big(2000(0.084373)+1000(0.285627)\big)\left(\frac{1-e^{-0.04}}{0.04}\right)=\boxed{\mathbf{445.41}}$$

(b) The forces of mortality are $\mu_{x+s}^{(1)}=\frac{0.1}{1-0.1s}$ and $\mu_{x+s}^{(2)}=\frac{0.3}{1-0.3s}$. Also, ${}_sp_x^{(\tau)}=(1-0.1s)(1-0.3s)$. So the EPV of the insurance is

$$\mathrm{EPV}=\int_0^1 v^s(1-0.1s)(1-0.3s)\left(2000\frac{0.1}{1-0.1s}+1000\frac{0.3}{1-0.3s}\right)\mathrm{d}s$$

$$
\begin{aligned}
&= \int_0^1 e^{-0.04s}(500-90s)\,\mathrm{d}s \\
&= -\frac{e^{-0.04s}}{0.04}(500-90s)\bigg|_0^1 - 90\frac{\int_0^1 e^{-0.04s}\,\mathrm{d}s}{0.04} \\
&= \frac{500-410e^{-0.04}}{0.04} - \frac{90(1-e^{-0.04})}{0.04^2} = \boxed{\mathbf{446.31}}
\end{aligned}
$$

(c) The forces of decrement are $-\ln p_x'^{(j)}$, or $\mu_x^{(1)} = -\ln 0.9$ and $\mu_x^{(2)} = -\ln 0.7$. The probability of survival from both decrements under constant force is

$$ {}_sp_x^{(\tau)} = {}_sp_x'^{(1)},\,{}_sp_x'^{(2)} = (0.9^s)(0.7^s) = 0.63^s $$

The EPV of the insurance is

$$
\begin{aligned}
\mathrm{EPV} &= \int_0^1 v^s\,{}_sp_x^{(\tau)}\,(2000\mu_x^{(1)} + 1000\mu_x^{(2)})\,\mathrm{d}s \\
&= \int_0^1 e^{(-0.04+\ln 0.63)s}\underbrace{(-2000\ln 0.9 - 1000\ln 0.7)}_{567.396}\mathrm{d}s \\
&= 567.396\int_0^1 e^{(-0.04+\ln 0.63)s}\,\mathrm{d}s \\
&= \frac{567.396}{-\ln 0.63 + 0.04}(1-0.63e^{-0.04}) = \boxed{\mathbf{446.09}}
\end{aligned}
$$

4. [Lesson ‘20]

(a) First let's calculate the net single premium. We can ignore the 100 per year factor; it just scales up the numbers.

$$
\begin{aligned}
\bar{A}_{60} &= \frac{1-e^{-0.05(40)}}{0.05(40)} = 0.432332 \\
\bar{a}_{60} &= \frac{1-0.432332}{0.05} = 11.35335
\end{aligned}
$$

$\bar{a}_{\overline{T}|} = \bar{a}_{60}$ when:

$$
\begin{aligned}
\frac{1-e^{-0.05t}}{0.05} &= \frac{1-0.432332}{0.05} \\
e^{-0.05t} &= 0.432332 \\
t &= -\frac{\ln 0.432332}{0.05} = 16.77121
\end{aligned}
$$

The probability that $T_{60} > 16.77121$ is $1 - 16.77121/40 = \boxed{\mathbf{0.58072}}$.

(b) First let's calculate the net single premium.

$$
\begin{aligned}
\bar{A}^{\,1}_{60:\overline{10}|} &= \frac{1-e^{-0.05(10)}}{0.05(40)} = 0.196735 \\
1000\bar{A}^{\,1}_{60:\overline{10}|} + 100\ddot{a}_{60} &= 196.735 + 1135.335 = 1332.070
\end{aligned}
$$

The present value of payments may be higher than 1332.070 in the first 10 years. However, let's begin by calculating the time $t > 10$ at which the present value of payments is higher than 1332.070.

$$\begin{aligned}100\left(\frac{1-e^{-0.05t}}{0.05}\right) &= 1332.070\\ e^{-0.05t} &= 0.333965\\ t &= -\frac{\ln 0.333965}{0.05} = 21.93438\end{aligned}$$

Now let's determine the time $t < 10$ for which the present value of payments is 1332.070.

$$\begin{aligned}1000e^{-0.05t} + 100\left(\frac{1-e^{-0.05t}}{0.05}\right) &= 1332.07\\ -1000e^{-0.05t} + 2000 &= 1332.07\\ e^{-0.05t} &= 0.667930\\ t &= -\frac{\ln 0.667930}{0.05} = 8.071437\end{aligned}$$

Note that the present value of payments increases during the first 10 years. You see this from the second line above; $e^{-0.05t}$ has a negative coefficient and is a decreasing function of t, so the left side of the equation increases as t increases. Thus the present value of payments is greater than 1332.07 in the ranges $(8.071437, 10]$ and $(21.93438, \infty)$. The probability that death occurs in one of those ranges is $\big((10-8.071437)+(40-21.93438)\big)/40 = \boxed{\mathbf{0.49986}}$.

(c) For death right after time 10, the present value of the payments is

$$100\bar{a}_{\overline{10}|} = 100\left(\frac{1-e^{-0.5}}{0.05}\right) = 786.94$$

For death at time $t \le 10$, the present value of the payments is $2000 - 1000e^{-0.05t}$, which is always greater than 786.94. Therefore, $\boxed{\mathbf{786.94}}$ is the minimum loss.

5. [Section 22.2 and Lesson 26]

(a) The insurance can be expressed as a level whole life insurance of 9000, plus a 10-year increasing term insurance of 1000, plus a 10-year deferred insurance of 11,000. See figure A.1. Let A be the net single premium for the insurance payable at the end of the year of death.

$$\begin{aligned}A &= 9000A_{35} + 1000(IA)^{1}_{35:\overline{10}|} + 11{,}000\,{}_{10}E_{35}\,A_{45}\\ &= 9(128.72) + 107.98 + 11(0.54318)(201.20) = 2468.63\end{aligned}$$

Multiplying by i/δ, we get $1.02971(2468.63) = \boxed{\mathbf{2541.97}}$.

(b)

$$\begin{aligned}\mu_{35} &\approx -0.5\ln(l_{36}/l_{34}) = -0.5\ln(9{,}401{,}688/9{,}438{,}571) = 0.0019577\\ 12\ddot{a}^{(12)}_{35} &= 12\left(15.3926 - \tfrac{11}{24} - \tfrac{143}{1728}(0.0019577 + \ln 1.06)\right) = \boxed{\mathbf{179.15}}\end{aligned}$$

(c) $2541.97/179.15 = \boxed{\mathbf{14.1889}}$.

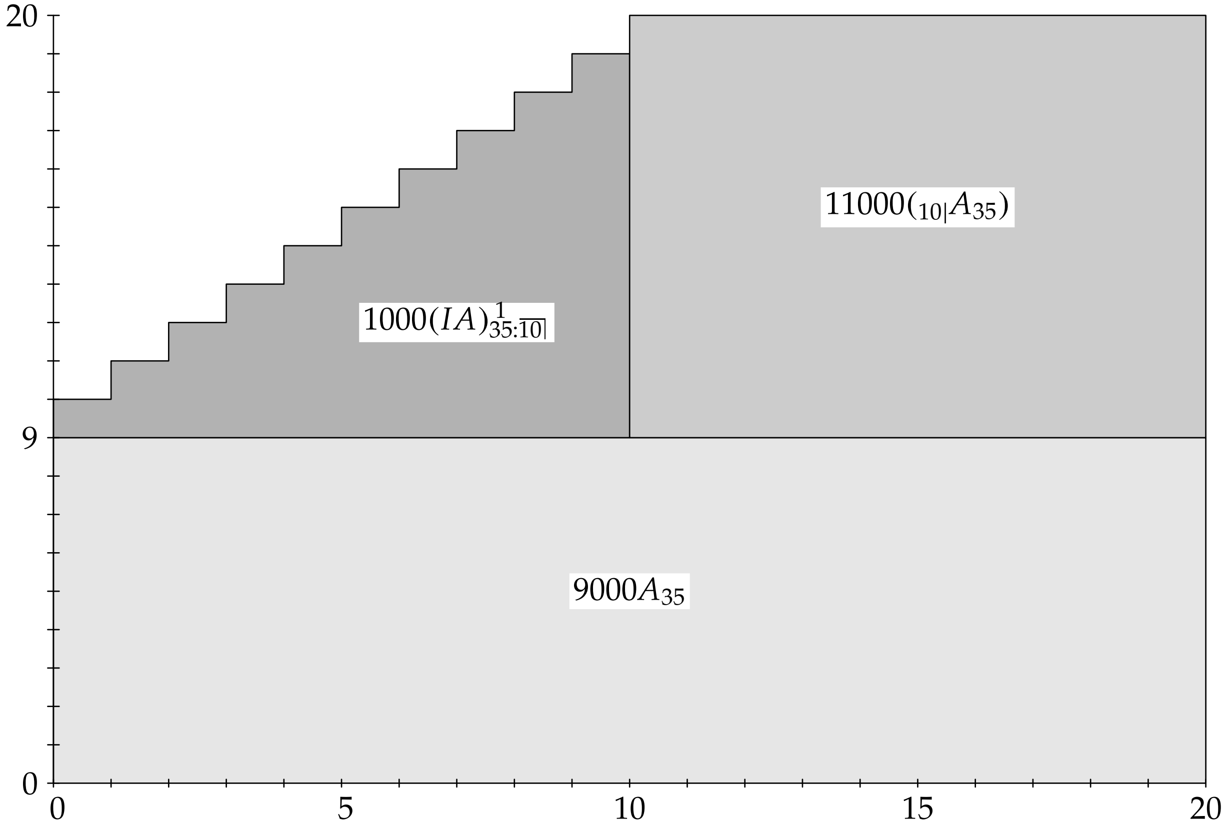


Figure A.1: Decomposition of increasing insurance in question 5

(d) We need to calculate $20{,}000\bar{A}_{45}$ and $\ddot{a}^{(12)}_{45}$.

$$\begin{aligned}
20{,}000\bar{A}_{45} &= 1.02971(20)(201.20) = 4143.55\\
\mu_{45} &\approx -0.5\ln(l_{46}/l_{44}) = -0.5\ln(9{,}127{,}426/9{,}198{,}149) = 0.0038592\\
12\ddot{a}^{(12)}_{45} &= 12\left(14.1121 - \tfrac{11}{24} - \tfrac{143}{1728}(0.0038592 + \ln 1.06)\right) = 163.78\\
{}_{10}V &= 4143.55 - 14.1889(163.78) = \boxed{\mathbf{1819.64}}
\end{aligned}$$

(e) This insurance can be decomposed into a 10-year decreasing insurance plus a 10-year deferred whole life insurance. The EPV of the decreasing insurance can be derived from

$$(IA)^{1}_{35:\overline{10}|} + (DA)^{1}_{35:\overline{10}|} = 11A^{1}_{35:\overline{10}|}$$

Let A be the net single premium for the insurance payable at the end of the year of death.

$$\begin{aligned}
A &= 1000\left(11A^{1}_{35:\overline{10}|} - (IA)^{1}_{35:\overline{10}|}\right) + 1000{}_{10}E_{35}\,A_{45}\\
&= 1000\Big(11\big(0.12872 - (0.54318)(0.20120)\big) - 0.10798\Big) + 1000(0.54318)(0.20120)\\
&= 215.06
\end{aligned}$$

Multiplying by i/δ, we get $1.02971(215.06) = \boxed{\mathbf{221.45}}$.

An equivalent alternative is to evaluate the insurance as a whole life insurance for 11,000 minus a 10-year term increasing insurance for 1000 minus a 10-year deferred whole life insurance for 10,000.

6. [Lesson 61]

(a) For Cramer, ${}_{10}E_{55} = 0.48686$.

$$20\big(0.015(100{,}000) + 0.02(20{,}000)\big)(0.48686)(11) = 203{,}507.5$$

For Liu, ${}_{30}E_{35} = (0.54318)(0.25634) = 0.13924$.

$$5\big((0.015)(50{,}000)\big)(0.13924)(11) = 5743.6$$

For Smith, ${}_{15}E_{50} = (0.72137)(0.48686) = 0.35121$.

$$10(0.015)(100{,}000)(0.35121)(11) = 57{,}949.7$$

Total actuarial liability is $203{,}507.5 + 5743.6 + 57{,}949.7 = \boxed{\mathbf{267{,}201}}$.

(b) For Cramer, salary will be $120,000(1.03) = 123{,}600$ next year. The discounted value of next year's liability is

$$21\big(0.015(100{,}000) + 0.02(23{,}600)\big)(0.48686)(11) = 221{,}780.3$$

For Liu, salary will not exceed 100,000, so we can calculate the normal contribution directly using formula (61.4):

$$5743.6\left(1.03\left(\frac{6}{5}\right) - 1\right) = 1355.5$$

For Smith, salary will be $100{,}000(1.03) = 103{,}000$ next year. The discounted value of next year's liability is

$$11\big(0.015(100{,}000) + 0.02(3{,}000)\big)(0.35121)(11) = 66{,}294.4$$

The normal contribution is $(221{,}780.3 - 203{,}507.5) + 1355.5 + (66{,}294.4 - 57{,}949.9) = \boxed{\mathbf{27{,}973}}$.

(c) Final salary is $120{,}000(1.03^{10}) = 161{,}270$. Annual pension is

$$30\big(0.015(100{,}000) + 0.02(61{,}270)\big) = 81{,}762$$

The replacement ratio is $81{,}762/161{,}270 = \boxed{\mathbf{0.5070}}$.

(d) Final salary is $100{,}000(1.03^{15}) = 155{,}797$. The annual payment under a monthly annuity-due is

$$25\big(0.015(100{,}000) + 0.02(55{,}797)\big) = 65{,}398$$

By Woolhouse's formula to two terms, $\ddot{a}_{65}^{(12)} = \ddot{a}_{65} - \frac{11}{24}$, so $\ddot{a}_{65} = 11\frac{11}{24}$, and

$$\begin{aligned} 63{,}130\ddot{a}_{65}^{(12)} &= x\ddot{a}_{65} \\ x &= 65{,}398\left(\frac{11}{11\frac{11}{24}}\right) = \boxed{\mathbf{62{,}782}} \end{aligned}$$

(e) This change only affects Liu. We must recalculate ${}_{30}E_{35}$ for Liu. We'll calculate it from first principles, although you may also calculate ${}_{5}E_{35}$ and then multiply by ${}_{25}E_{40}$ which can be calculated from the pure endowment columns of the Illustrative Life Table.

$${}_{30}p_{35} = {}_{4}p_{35}\, p_{39}\, {}_{25}p_{40}$$

$$= \left(\frac{9{,}337{,}427}{9{,}420{,}657}\right)(1 - 0.00244)\left(\frac{7{,}533{,}964}{9{,}313{,}166}\right) = 0.799855$$

$$ {}_{30}E_{35} = \frac{0.799855}{1.06^{30}} = 0.13926$$

The revised liability for Liu is

$$5\big((0.015)(50{,}000)\big)(0.13926)(11) = 5744.6$$

instead of the previous 5743.6 calculated in part (a). The actuarial liability increases by 1 and becomes $\boxed{\textbf{267,202}}$.