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# Exam P Study Manual

## 6th Edition

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

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# Exam P Study Manual

**6th Edition**

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

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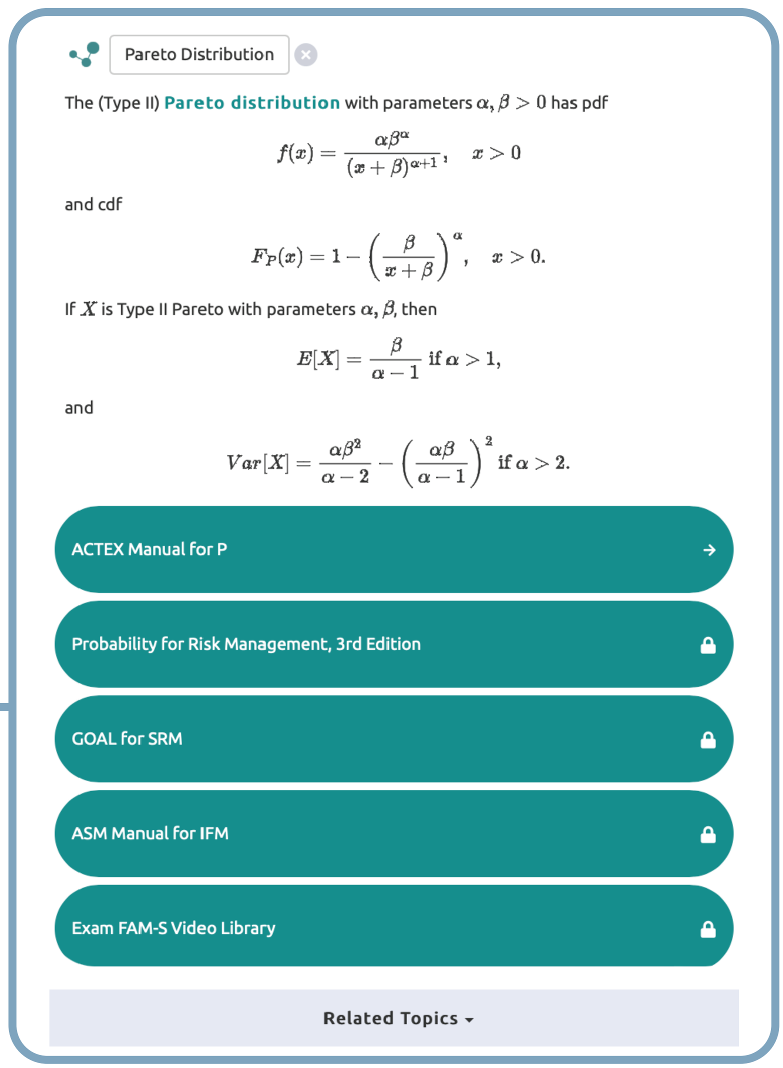

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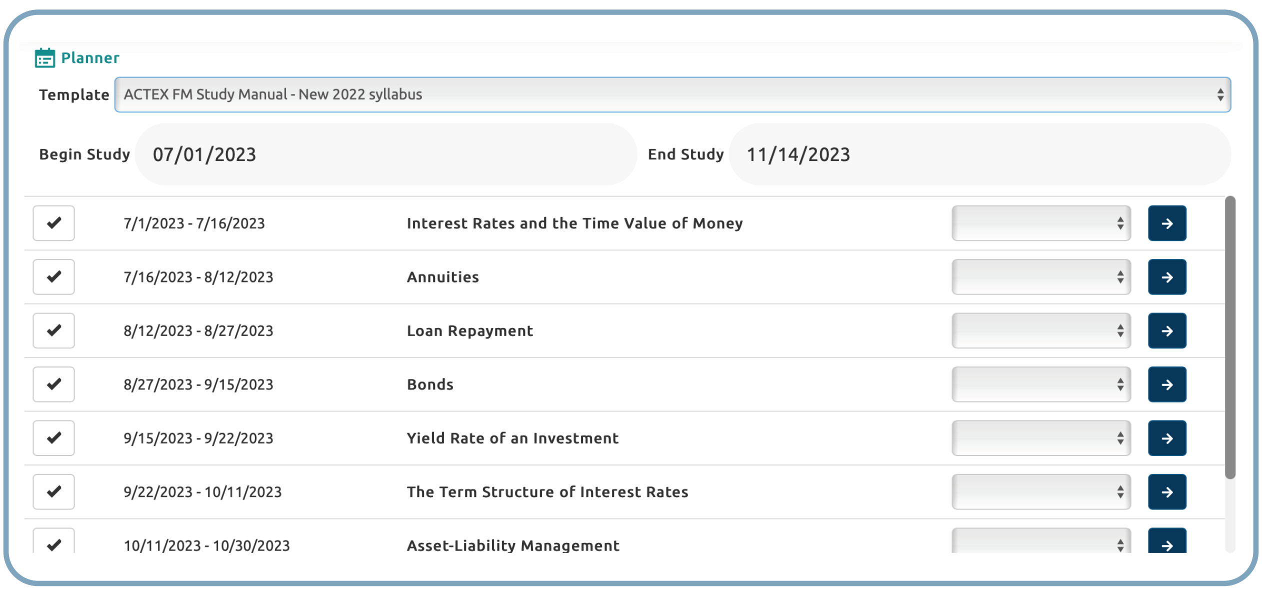
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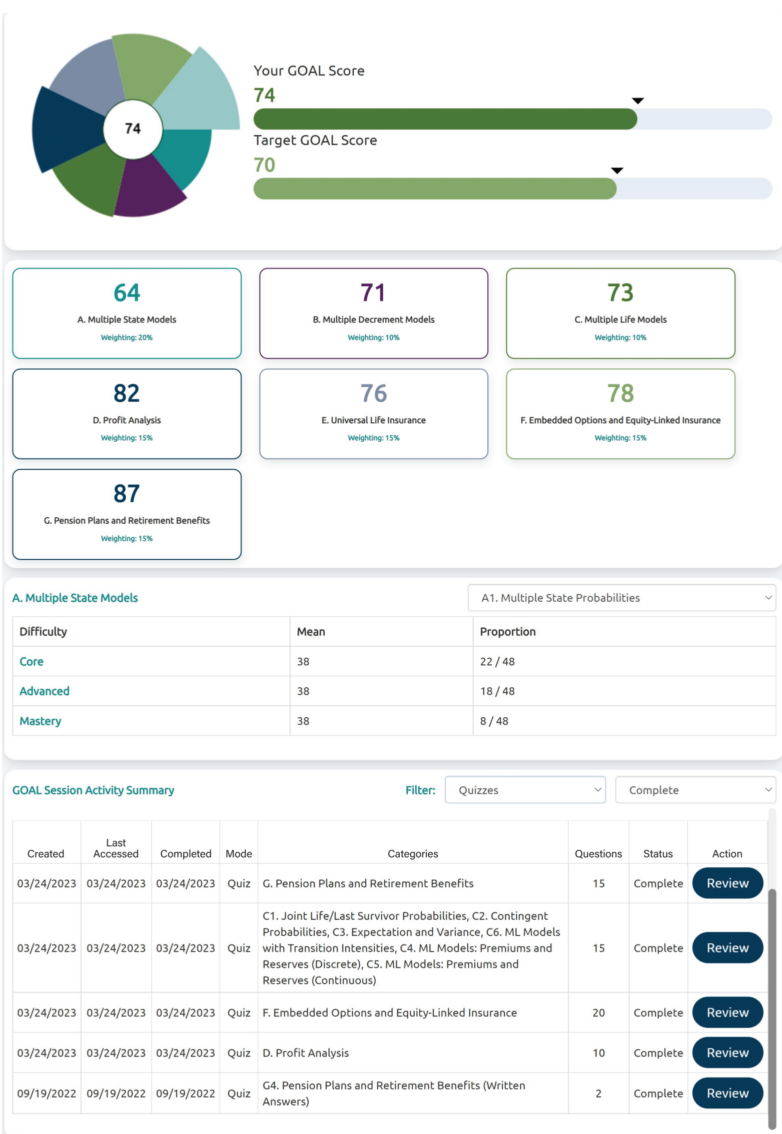


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# Contents

# Introduction

Welcome to Exam P! This manual will prepare you for the exam. You are given 23 bite-size chunks, each with explanations, techniques, examples, and lots of exercises. If you learn the techniques in these 23 lessons, you will be well-prepared for the exam.

A prerequisite to this exam is calculus. Heavy calculus techniques are not tested, but you must know the basics. A short review of integration techniques follows this introduction. If these look totally unfamiliar to you, you must review your calculus before proceeding further!

## Exercises in this manual

There are several types of exercises in this manual:

1. Questions from the SOA 485.[1] These are probably the most representative of what will appear on your exam. The lower numbered ones come from the exams released between 2000 and 2003. The higher numbered ones are probably retired questions from the CBT (computer based testing) data bank, the question bank from which the questions on your CBT exam are taken from.

2. Questions from the five released exams 2000–2003 not contained in the SOA 485. Those exams tested both probability and calculus, so only about half the questions are relevant. Almost all of the relevant questions are in the SOA 485, but a small number of them are not in that list.

3. Questions from the 1999 Sample Exam. This sample exam was published before the first exam under the 2000 syllabus. The 2000 syllabus was a drastic syllabus change and was accompanied by a new exam question style. This sample exam reflects the new style of question, but most of the questions never appeared on a real exam, so the questions may not be totally realistic exam questions.

4. Questions from pre-2000 released exams. The probability syllabus was hardly different before 2000—it is extremely stable—but the style of questions was much different. Questions before 2000 tended to be stated purely mathematically, whereas starting in 2000 almost all questions have a practical sounding context. Thus a pre-2000 exam question might be:

   $A$ and $B$ are two events. You are given that $P[A] = 0.8$, $P[B] = 0.6$, and $P[A \cup B] = 0.9$.

   What is $P[A \cap B]$?

   The same question appearing on a 2000 or later exam would read:

   A survey of cable TV subscribers finds:
   - 80% of subscribers watch the Nature channel.
   - 60% of subscribers watch the History channel.
   - 90% of subscribers watch at least one of the Nature or History channels.

   Calculate the percentage of subscribers that watch both the Nature and the History channels.

   Notice, among other things, that the final line of the question is no longer a question; it is always a directive. Some of the post-2000 released exam questions still asked questions. When these questions were incorporated into the SOA 485, they changed the question to a directive. Questions are no longer used.

   I was thinking of rewording all the pre-2000 questions in the current style. But I am not good at creating contexts. I'd probably just copy contexts from existing questions, and it would get boring. Even though the style of these old questions is different, they are based on the same material, so working them out will help you learn the material you need to know for this exam.

[1] Since 15 questions are deleted, this list really contains 470 questions.

5. A small number of relevant questions from other exams, mostly in Lesson 23.
6. Original questions. I try to write my questions in the current style, but I often copy contexts from SOA 485 questions.

The solutions to exercises are not necessarily the best methods for solving them. Sometimes, a technique in a later lesson may provide a shortcut. This is particularly true for exercises in the earlier lessons using standard distributions as illustrations. For example, an exercise in an early lesson may ask for the mean of a conditional distribution involving an exponential; after learning in the exponential lesson that an exponential is memoryless, this sort of question may be trivial. *However, if you find a better way to do an exercise that does not involve techniques from later lessons, please contact the author at the same address as the one for sending errata.*

Adding up the SOA sample questions, other old exam questions, and 111 original questions, there are 727 questions distributed among the 23 lessons. That's an average of more than 30 questions per lesson! You don't have to do them all, but you should aim to get lots of practice, enough so that working out these questions becomes second nature.

This manual has six practice exams. All the questions on these practice exams are original.

## Useful features of this manual

As noted above, there is a brief calculus note right after this introduction, for reviewing integration techniques.

There is an index at the end of the manual. If you ever remember a term and don't remember where you saw it, refer to the index. If it isn't in the index and you think it is in the manual, contact the author so that he can add it to the index.

Before the index, there are three cross reference tables. The first table is a cross reference for the practice exams. Some students prefer to do the practice exam questions as additional practice after each lesson, rather than saving them for final review, and this cross reference table lets you do that.

The second and third tables show you where each of the SOA 485 questions is listed as an exercise in the manual. If you are interested in seeing my solutions to these questions, these tables are the place to go.

## SOA downloads

On the SOA website, you will find the syllabus for the exam. This syllabus has links to other useful material. You'll find the SOA 485 questions and solutions.[2]

There is an important link to *Risk and Insurance* study note, authored by Judy Feldman Anderson and Robert L. Brown. At this writing, the URL of the study note is `https://www.soa.org/globalassets/assets/files/edu/P-21-05.pdf`. This note gives you background information on how insurance works. You will not be tested directly on this study note, but insurance is used very often as the context of a question, so you should understand basic insurance terminology.

This study note has some probability examples, so you won't understand it in its entirety if you have never studied probability. Still, you should read the nonmathematical parts (in bed, or at some other leisure time) as soon as you can, ignoring the mathematical examples, since the exercises in this manual often have insurance contexts. At the point indicated in the manual in Lesson 8, you will have all the probability background you need to understand the study note's mathematics, and should read the study note in its entirety.

Another important link is to the table of the cumulative distribution function of the standard normal distribution. At this writing, it is at `https://www.soa.org/globalassets/assets/files/edu/2021/p-1-table-rev-4-29-21.pdf`. You do not need this table until Lesson 21, but you should download it by then. For your convenience, a standard normal distribution table is provided in Appendix B.

[2] I wrote my solution without looking at theirs. Usually they're the same, but the different wording may help you. In a few cases, my solution is better.

## Syllabus

The syllabus splits the material into three broad topics. The following table shows the split and the lessons corresponding to the split.

| Topic | Weight | Lessons |
|---|---|---|
| General probability | 23–30% | 1–4 |
| Univariate random variables | 44–50% | 5–10, 15–22 |
| Multivariate random variables | 23–30% | 11–14, 23 |

The part of the manual labeled "Supplementary Topics" are non-syllabus topics. These are topics that were on the syllabus before September 2022. You do not need to study these; they are included for professors who wish to teach them. However, some of the material may be useful for later exams. In particular, knowledge of moment and probability generating functions will be useful for Exam FAM.

## Errata

Please report any errors you find. Reports may be sent to the publisher (mail@studymanuals.com) or directly to me (errata@aceyourexams.net). *When reporting errata, please indicate which manual and which edition and printing you are referring to!* This manual is the 6th edition of the Exam P manual.

An errata list will be posted at http://errata.aceyourexams.net

## Acknowledgements

I thank the Society of Actuaries and the Casualty Actuarial Society for permission to use their old and sample exam questions. These questions are the backbone of this manual.

I thank the publisher, SR Books, for all the work they did to incorporate this manual into a modern online educational system that allows you to get additional practice working out questions written by other authors.

I thank Geoff Tims for proofreading the manual. As a result of his work, many mathematical errors were corrected and many unclear passages were clarified. I also thank Kristen McLaughlin for her comments and suggestions for improvements.

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# Calculus Notes

The exam will not test deep calculus. You will be expected to know how to differentiate and integrate polynomials, simple rational functions, logarithms, and exponentials, but trigonometric functions and integrals will rarely appear. Fancy integration techniques are not needed.

We will review logarithmic differentiation and basic integration techniques.

**Logarithmic differentiation** When a function $f(x)$ is a product or quotient of several functions, and you need to evaluate its derivative at a point, it is often easier to use logarithmic differentiation rather than to differentiate the function directly. Logarithmic differentiation is based on the formula for the derivative of a logarithm:

$$\frac{\mathrm{d}\ln f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f(x)/\mathrm{d}x}{f(x)}$$

It follows that

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = f(x)\left(\frac{\mathrm{d}\ln f(x)}{\mathrm{d}x}\right)$$

In other words, the derivative of a function is the function times the derivative of its logarithm. If the function is a product or quotient of functions, its logarithm will be a sum or difference of logarithms of those functions, and that will be easier to differentiate.

**Partial fraction decomposition** You should know that $x/(1+x) = 1 - 1/(1+x)$, for example, so that

$$\int_0^3 \frac{x\,\mathrm{d}x}{1+x} = \int_0^3 \left(1 - \frac{1}{1+x}\right)\mathrm{d}x = 3 - \ln(1+x)\Big|_0^3 = 3 - \ln 4$$

Any partial fraction decomposition more complicated is unlikely to appear.

This particular integral can also be evaluated using substitution, as we will now discuss.

**Substitution** There are two types of substitution.

The first type assists you to identify an antiderivative. Suppose you have

$$\int_0^3 xe^{-x^2/2}\,\mathrm{d}x$$

You may realize that the integrand is negative the derivative of $e^{-x^2/2}$. When you differentiate $e^{-x^2/2}$, by the chain rule, you multiply by the derivative of $-x^2/2$, which is $-x$, and obtain $-xe^{-x^2/2}$. So the integral is $-e^{-x^2/2}$ evaluated at 3 minus the same evaluated at 0. But if you didn't recognize this integrand, you could substitute $y = -x^2/2$ and get:

$$\begin{aligned}
\mathrm{d}y &= -x\,\mathrm{d}x\\
x = 0 &\Rightarrow y = 0\\
x = 3 &\Rightarrow y = -4.5\\
\int_0^3 xe^{-x^2/2}\,\mathrm{d}x &= \int_0^{-4.5} -e^y\,\mathrm{d}y\\
&= -e^y\Big|_0^{-4.5}\\
&= 1 - e^{-4.5}
\end{aligned}$$

The second type simplifies an integral or makes it doable. We evaluated $\int_0^3 \frac{x\,dx}{1+x}$ above using partial fraction decomposition. An alternative would be to set $y = 1 + x$, $dy = dx$. The bounds of the integral become 1 and 4, since $0 + 1 = 1$ and $3 + 1 = 4$, and we get

$$\begin{aligned}\int_0^3 \frac{x\,dx}{1+x} &= \int_1^4 \frac{(y-1)\,dy}{y}\\ &= \int_1^4 \left(1 - \frac{1}{y}\right) dy\\ &= \left(y - \ln y\right)\Big|_1^4\\ &= (4-1) + (\ln 4 - \ln 1) = 3 - \ln 4\end{aligned}$$

As another example, suppose you have

$$\int_0^1 x(1-x)^8 dx$$

You could expand $(1-x)^8$, multiply it by $x$, and integrate 9 terms. But it is easier to substitute $y = 1 - x$ and get

$$\begin{aligned} dy &= -dx\\ x = 0 &\Rightarrow y = 1\\ x = 1 &\Rightarrow y = 0\\ \int_0^1 x(1-x)^8 dx &= \int_1^0 -(1-y)y^8\,dy\\ &= \int_0^1 (y^8 - y^9) dy\\ &= \frac{1}{9} - \frac{1}{10} = \frac{1}{90}\end{aligned}$$

Another technique to evaluate this integral is integration by parts, differentiating $x$ and integrating $(1-x)^8$.

**Integration by parts** When integrating by parts, we use $\int u\,dv = uv - \int v\,du$. We integrate one expression and evaluate its product with the other, then differentiate the other and evaluate the integral with the derivative and the integrated expression.

If you are given

$$\int_0^1 x(1-x)^8 dx$$

set $u = x$ and $dv = (1-x)^8\,dx$ and you get

$$\begin{aligned}\int_0^1 x(1-x)^8 dx &= -\frac{x(1-x)^9}{9}\Big|_0^1 + \int_0^1 \frac{(1-x)^9}{9} dx\\ &= -\left.\frac{(1-x)^{10}}{90}\right|_0^1 = \frac{1}{90}\end{aligned}$$

A technique that is helpful, especially if integration by parts has to be repeated, is tabular integration. Set up a table. Each row has two columns. The first row has the two expressions, the one you want to differentiate (left column) and the one you want to integrate (right column). On each successive row, differentiate the left column entry from the previous row and integrate the right column entry from the previous row. Keep doing this until the left column entry is 0. Then the antiderivative is the alternating sum of the products of the $k^{\text{th}}$ entry of the left column and the $k+1^{\text{st}}$ entry of the second column, $k = 1, 2, \ldots$ up to but not including the final 0 in the left column. The sign of each summand is $(-1)^{k-1}$.

For example, suppose you want to evaluate

$$\int_0^3 x^2 e^{x/5} \mathrm{d}x$$

The table looks like this:

| | | |
|---|---|---|
| $+$ | $x^2$ | $e^{x/5}$ |
| $-$ | $2x$ | $5e^{x/5}$ |
| $+$ | $2$ | $25e^{x/5}$ |
| | $0$ | $125e^{x/5}$ |

The entries connected with a line are multiplied, and added or subtracted as indicated by the sign. The result is

$$\int_0^3 x^2 e^{x/5} \mathrm{d}x = \left(5x^2 e^{x/5} - 50xe^{x/5} + 250e^{x/5}\right)\Big|_0^3 = 14.20723$$

We often integrate $xe^{-ax}$. If you wish, you may memorize the following results, especially the first one, so that you don't have to integrate by parts each time you need them:

$$\int_0^\infty xe^{-ax} \,\mathrm{d}x = \frac{1}{a^2} \quad \text{for } a > 0 \tag{1}$$

$$\int_0^\infty x^2 e^{-ax} \,\mathrm{d}x = \frac{2}{a^3} \quad \text{for } a > 0 \tag{2}$$

# Lesson 8

# Variance and other Moments

The concept of expected value of a random variable can be generalized. We may compute the expected value of a function of a random variable, $\mathbf{E}[g(X)]$. This is defined as

- $\sum g(x)p(x)$ for a discrete random variable
- $\int g(x)f(x)\,dx$ for a continuous random variable

Some important examples of $\mathbf{E}[g(X)]$ are

- $\mathbf{E}[X^k]$ is called the *$k^{th}$ (raw) moment* of $X$. The mean is the first moment, and is denoted by $\mu$.
- Let $\mu = \mathbf{E}[X]$. Then $\mathbf{E}[(X-\mu)^k]$ for $k \neq 1$ is called the *$k^{th}$ central moment* of $X$.
- The second central moment of $X$, $\mathbf{E}[(X-\mu)^2]$, is called the *variance*. It is denoted by $\operatorname{Var}(X)$ and by $\sigma^2$. It measures the dispersion of the distribution. We will discuss variance in detail in this lesson.
- $\sigma$, the positive square root of $\sigma^2$, is called the *standard deviation*.
- The coefficient of variation is defined as the ratio of the standard deviation to the mean, $\sigma/\mu$. It is a unitless measure, since the units of the standard deviation are canceled by the units of the mean. It is a measure of the variability of the random variable. It is not defined in most probability textbooks and is used mostly in insurance contexts.
- $\dfrac{\mathbf{E}[(X-\mu)^3]}{\sigma^3}$ is the coefficient of *skewness*. It is a unitless measure. It measures the extent to which a distribution has values further away from the mean on the right (higher than the mean), in which case skewness is positive, or on the left (lower than the mean), in which case skewness is negative. We will sometimes call this skewness, that is, we'll leave out "coefficient of".
- $\dfrac{\mathbf{E}[(X-\mu)^4]}{\sigma^4}$ is the coefficient of *kurtosis*. It is a unitless measure. It measures the extent to which a distribution's density drops from its peak quickly (low kurtosis) or slowly (high kurtosis). We will sometimes call this kurtosis, that is, we'll leave out "coefficient of".

**Example 8A**

A random variable is equal to 0 with probability 1/2 and 1 with probability 1/2.
Calculate the random variable's kurtosis.

**Solution:** The mean is $\mu = 0.5(0) + 0.5(1) = 0.5$. The variance is

$$\begin{aligned}\sigma^2 &= \mathbf{E}[(X-\mu)^2]\\ &= \sum_{x=0}^{1} p(x)(x-\mu)^2\\ &= 0.5(0-0.5)^2 + 0.5(1-0.5)^2 = 0.25\end{aligned}$$

The fourth central moment is

$$\begin{aligned}\mathbf{E}[(X-0.5)^4] &= \sum_{x=0}^{1} p(x)(x-\mu)^4\\ &= 0.5(0-0.5)^4 + 0.5(1-0.5)^4 = 0.0625\end{aligned}$$

Kurtosis is $0.0625/0.25^2 = \boxed{1}$. □

Expectation is a linear operator. What this means is that for two functions $g(X)$ and $h(X)$ of $X$,

$$\mathbf{E}[g(X) + h(X)] = \mathbf{E}[g(X)] + \mathbf{E}[h(X)]$$

It also means that for any constant $c$,

$$\mathbf{E}[cg(X)] = c\,\mathbf{E}[g(X)]$$

**Example 8B**

You are given

i) $\mathbf{E}[X] = 10$

ii) $\mathbf{E}[X^2] = 150$

Calculate $\mathbf{E}[(X-5)^2]$.

**Solution:**

$$\mathbf{E}[(X-5)^2] = \mathbf{E}[X^2 - 10X + 25] = \mathbf{E}[X^2] - 10\,\mathbf{E}[X] + 25 = 150 - 10(10) + 25 = \boxed{75}$$ □

With this linearity property, we develop an alternative formula for variance:

$$\mathrm{Var}(X) = \mathbf{E}[(X-\mu)^2] = \mathbf{E}[X^2 - 2\mu X + \mu^2] = \mathbf{E}[X^2] - 2\mu\,\mathbf{E}[X] + \mu^2$$

However, $\mathbf{E}[X] = \mu$, so we can write this as

**Variance Formula**

$$\mathrm{Var}(X) = \mathbf{E}[X^2] - \mu^2 \tag{8.1}$$

This formula is the one usually used to calculate variance.

**Example 8C**

A discrete random variable has the following probability function:

| Value | 1 | 3 | 4 | 7 |
|---|---|---|---|---|
| Probability | 0.55 | 0.20 | 0.15 | 0.10 |

Calculate the variance of this random variable.

**Solution:** Let $X$ be the random variable.

$$\mathbf{E}[X] = 0.55(1) + 0.20(3) + 0.15(4) + 0.10(7) = 2.45$$
$$\mathbf{E}[X^2] = 0.55(1^2) + 0.20(3^2) + 0.15(4^2) + 0.10(7^2) = 9.65$$
$$\mathrm{Var}(X) = 9.65 - 2.45^2 = \boxed{3.6475}$$

This calculation can be done using the statistics registers of the TI-30XS Multiview calculator. Enter the values in one column and the probabilities in another column of the data table. Then ask for 1-variable statistics with the frequencies equal to the probabilities. The sigma register (#4) is the square root of the variance. Register #5 is $\mathbf{E}[X]$ and register #6 is $\mathbf{E}[X^2]$. □

While many calculators can calculate the variance of discrete random variables with a finite number of possible values, you will find formula (8.1) useful for continuous and mixed random variables.

Variance has the following multiplication property:

$$\mathrm{Var}(cX) = c^2\,\mathrm{Var}(X)$$

Adding a constant to a random variable has no effect on its variance. So we can extend this multiplication formula as follows:

$$\mathrm{Var}(aX + b) = a^2 \,\mathrm{Var}(X) \tag{8.2}$$

Variance has an additive property. If $X_i$, $i = 1, \ldots, n$ are mutually independent random variables, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

In particular, if all of the $X_i$ are mutually independent and have the same distribution $X$, and $Y = \sum_{i=1}^{n} X_i$, then

$$\mathrm{Var}(Y) = n\,\mathrm{Var}(X)$$

The additive property applies only for *independent* random variables. We will discuss variance of sums of non-independent random variables in Lesson 13.

At this point, you have the mathematical background to understand the Risk and Insurance study note. I recommend you read it now for background on how insurance works.

## 8.1 Bernoulli shortcut

Later on in the course we'll study some standard distributions. Right now, let's consider a simple discrete distribution, called the Bernoulli distribution. A Bernoulli random variable $N$ can only assume the values 0 and 1. It assumes the value 1 with probability $p$ and the value 0 with probability $1 - p$. What are the mean and variance of $N$?

$$\mathbf{E}[N] = 0(1 - p) + 1(p) = p$$

$$\mathrm{Var}(N) = (0 - p)^2(1 - p) + (1 - p)^2(p) = p(1 - p)(p + 1 - p) = p(1 - p)$$

The variance is the product of the probabilities of 0 and 1.

Let's generalize this. Let $X$ be a random variable that assumes only two values, $a$ and $b$. It assumes $b$ with probability $p$ and $a$ with probability $1 - p$. This is a linear function of a Bernoulli variable. Let $N$ be a Bernoulli variable. Then $X = a + (b - a)N$. What is the variance of $X$? By equation (8.2),

$$\mathrm{Var}(X) = (b - a)^2 p(1 - p) \tag{8.3}$$

The variance of $X$ is the square of the difference of values times the two probabilities. Remember this formula! It's very useful, and much faster than calculating first and second moments. It applies to any random variable that can assume only two values.

**Example 8D**

On good days you pass an exam with a 10 and on bad days you fail an exam with a 5. 70% of your days are good. What is the variance of your score?

**Solution:**

$$(0.7)(0.3)(10 - 5)^2 = \boxed{5.25}$$

□

## Exercises

**8.1.** **[Sample:328]** The operating cost of a new claims system is modeled by a random variable $X$ with variance 50. In its second year of use, inflation of 3% and an additional fixed maintenance cost of 2.5 increase the operating cost of the system.

Calculate the variance of the operating cost of the claims system in its second year of use.

(A) 52 (B) 53 (C) 54 (D) 56 (E) 59



*Exercises continue on the next page . . .*

**Table 8.1:** Summary of moments

| |
|---|
| **$k^{\text{th}}$ raw moment** $\mathbf{E}[X^k]$ |
| **$k^{\text{th}}$ central moment** $\mathbf{E}[(X-\mu)^k]$ |
| **Variance** $\sigma^2$ or $\mathrm{Var}(X)$. It equals $\mathbf{E}[(X-\mu)^2] = \mathbf{E}[X^2] - \mu^2$ |
| **Standard deviation** $\sigma = \sqrt{\mathrm{Var}(X)}$ |
| **Coefficient of variation** $\sigma/\mu$ |
| **Skewness** $\dfrac{\mathbf{E}[(X-\mu)^3]}{\sigma^3}$ |
| **Kurtosis** $\dfrac{\mathbf{E}[(X-\mu)^4]}{\sigma^4}$ |
| **Linearity of expectation** $\mathbf{E}[ag(X)+bh(X)] = a\,\mathbf{E}[g(X)] + b\,\mathbf{E}[h(X)]$ |
| **Variance of linear function** $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$ |
| **Bernoulli shortcut** If $\Pr(X=a) = 1-p$ and $\Pr(X=b) = p$, then $\mathrm{Var}(X) = (b-a)^2p(1-p)$. |

**8.2.** You are given that $\mathbf{E}[X] = 10$ and $\mathrm{Var}(X) = 40$.
Calculate $\mathbf{E}[(X+5)^2]$.

**8.3.** **[110-S83:30]** Figure A shows the probability density functions of two symmetric bounded distributions with the same median.

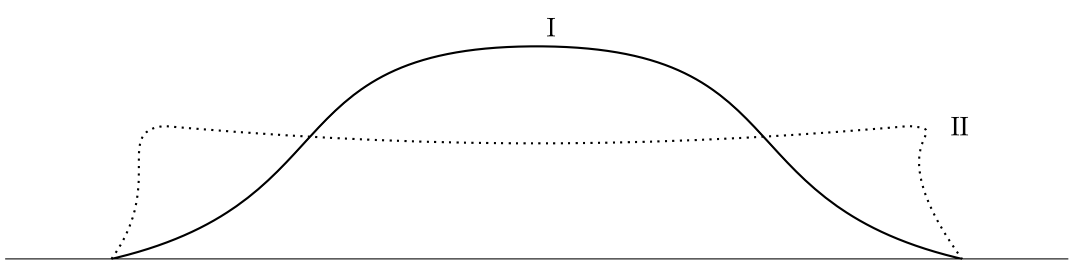



Figure A

Which of the following statements about the means and standard deviations of the two distributions are true?

(A) $\mu_{II} > \mu_I$ and $\sigma_I = \sigma_{II}$
(B) $\mu_{II} > \mu_I$ and $\sigma_I > \sigma_{II}$
(C) $\mu_I = \mu_{II}$ and $\sigma_{II} \le \sigma_I$
(D) $\mu_I = \mu_{II}$ and $\sigma_I \le \sigma_{II}$
(E) Cannot be determined from the given information.



Exercises continue on the next page . . .

**8.4.** **[110-S92:2]** Let $X$ be a random variable such that $\mathbf{E}[X] = 2$, $\mathbf{E}[X^3] = 9$, and $\mathbf{E}[(X-2)^3] = 0$.
What is $\text{Var}(X)$?

(A) $\frac{1}{6}$ (B) $\frac{13}{6}$ (C) $\frac{25}{6}$ (D) $\frac{49}{6}$ (E) $\frac{17}{2}$

**8.5.** **[F00:1,Sample:55]** A recent study indicates that the annual cost of maintaining and repairing a car in a town in Ontario averages 200 with a variance of 260.

A tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive).

Calculate the variance of the annual cost of maintaining and repairing a car after the tax is introduced.

(A) 208 (B) 260 (C) 270 (D) 312 (E) 374

**8.6.** **[S03:32,Sample:56]** A random variable $X$ has the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x^2 - 2x + 2}{2} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

Calculate the variance of $X$.

(A) $\dfrac{7}{72}$ (B) $\dfrac{1}{8}$ (C) $\dfrac{5}{36}$ (D) $\dfrac{4}{3}$ (E) $\dfrac{23}{12}$

**8.7.** **[S00:8,Sample:58]** A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

| Claim Size | Probability |
|---|---|
| 20 | 0.15 |
| 30 | 0.10 |
| 40 | 0.05 |
| 50 | 0.20 |
| 60 | 0.10 |
| 70 | 0.10 |
| 80 | 0.30 |

Calculate the percentage of claims that are within one standard deviation of the mean claim size.

(A) 45% (B) 55% (C) 68% (D) 85% (E) 100%

**8.8.** **[Sample:118]** An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. For every full ten inches of snow in excess of 40 inches during the winter season, the insurer pays the airport 300 up to a policy maximum of 700.

The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

| Inches | [0,20) | [20,30) | [30,40) | [40,50) | [50,60) | [60,70) | [70,80) | [80,90) | $[90,\infty)$ |
|---|---|---|---|---|---|---|---|---|---|
| Probability | 0.06 | 0.18 | 0.26 | 0.22 | 0.14 | 0.06 | 0.04 | 0.04 | 0.00 |

Calculate the standard deviation of the amount paid under the policy.

(A) 134 (B) 235 (C) 271 (D) 313 (E) 352



*Exercises continue on the next page . . .*

**8.9.** **[Sample:289]** An airport owner purchases an insurance policy to offset costs associated with excessive amounts of snowfall. For every full ten inches of snow in excess of 40 inches during the winter season, the insurer pays the airport 200 up to a policy maximum of 500.

The following table shows a probability function for the random variable $X$ of winter season snowfall, in inches, at the airport.

| **Inches of Snowfall** ($x$) | $p(x)$ |
|---|---|
| $0 \le x < 20$ | 0.06 |
| $20 \le x < 30$ | 0.18 |
| $30 \le x < 40$ | 0.26 |
| $40 \le x < 50$ | 0.22 |
| $50 \le x < 60$ | 0.14 |
| $60 \le x < 70$ | 0.06 |
| $70 \le x < 80$ | 0.04 |
| $80 \le x < 90$ | 0.04 |
| $90 \le x$ | 0.00 |

Calculate the standard deviation of the amount paid under the policy.

(A) 163.5 (B) 187.6 (C) 208.7 (D) 234.9 (E) 336.6

**8.10.** **[Sample:352]** A homeowner purchases flood insurance that pays a benefit based on the amount of rain that falls. No benefit is paid for rainfall amounts less than twelve inches. For every full two inches greater than twelve, the insurer pays the homeowner 5000, with a maximum payment of 18,000.

The following table displays probabilities for the rainfall amounts.

| Inches of Rain ($x$) | Probability of being in interval |
|---|---|
| $0 \le x < 2$ | 0.04 |
| $2 \le x < 4$ | 0.06 |
| $4 \le x < 6$ | 0.07 |
| $6 \le x < 8$ | 0.09 |
| $8 \le x < 10$ | 0.12 |
| $10 \le x < 12$ | 0.14 |
| $12 \le x < 14$ | 0.18 |
| $14 \le x < 16$ | 0.11 |
| $16 \le x < 18$ | 0.08 |
| $18 \le x < 20$ | 0.07 |
| $20 \le x$ | 0.04 |

Calculate the standard deviation of the benefit paid under the policy.

(A) 2201 (B) 3120 (C) 3200 (D) 5452 (E) 5680

**8.11.** **[Sample:233]** Ten cards from a deck of playing cards are in a box: two diamonds, three spades, and five hearts. Two cards are randomly selected without replacement.

Calculate the variance of the number of diamonds selected, given that no spade is selected.

(A) 0.24 (B) 0.28 (C) 0.32 (D) 0.34 (E) 0.41

**8.12.** **[Sample:232]** The number of claims $X$ on a health insurance policy is a random variable with $\mathbf{E}[X^2] = 61$ and $\mathbf{E}[(X-1)^2] = 47$.

Calculate the standard deviation of the number of claims.

(A) 2.18 (B) 2.40 (C) 7.31 (D) 7.50 (E) 7.81



*Exercises continue on the next page . . .*

**8.13.** **[Sample:294]** The probability that the economy will improve, remain stable, or decline is:

| State of the Economy | Probability |
|---|---|
| Improve | 0.30 |
| Remain stable | 0.50 |
| Decline | 0.20 |

Prices for Stock X and Stock Y will change as follows:

| State of the Economy | State X | State Y |
|---|---|---|
| Improve | Increase 18% | Increase 15% |
| Remain Stable | Increase 8% | Increase 7% |
| Decline | Decrease 13% | Decrease 6% |

Determine which of the following statements about the percentage price changes for Stock X and Stock Y is true.

(A) The percentage change for Stock X has a larger variance and a larger mean.
(B) The percentage change for Stock X has a larger variance and the means are equal.
(C) The percentage change for Stock X has a larger variance and a smaller mean.
(D) The variances are equal and the percentage change for Stock X has a larger mean.
(E) Both the variances and the means are equal.

**8.14.** The distribution of losses $X$ on an insurance policy has the following probability density function:

$$f(x) = \begin{cases} \dfrac{500}{x^2} & x > 500 \\ 0 & \text{otherwise} \end{cases}$$

The policy has a benefit limit of 2000.

Calculate the variance of the payment per loss on the policy.

**8.15.** The distribution function of a random variable $X$ is

$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{x}{c} & 0 \le x \le c \\ 1 & x > c \end{cases}$$

Calculate the coefficient of variation of $X$.

### Other moments, and variance of functions

**8.16.** A random variable $N$ has the following probability function:

$$P[n] = \begin{cases} 0.3 & n = 0 \\ 0.5 & n = 1 \\ 0.2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the skewness of $N$.



Exercises continue on the next page . . .

**8.17.** **[4B-S93:34]** (1 point) Claim severity has the following distribution:

| Claim Size | Probability |
|---|---|
| 100 | 0.05 |
| 200 | 0.20 |
| 300 | 0.50 |
| 400 | 0.20 |
| 500 | 0.05 |

Determine the distribution's skewness.

(A) −0.25 (B) 0 (C) 0.15 (D) 0.35
(E) Cannot be determined

**8.18.** **[4B-S97:21]** (2 points) You are given the following:

- Both the mean and the coefficient of variation of a particular distribution are 2.
- The third moment of this distribution about the origin is 136.

Determine the skewness of this distribution.

Hint: The skewness of a distribution is defined to be the third central moment divided by the cube of the standard deviation.

(A) 1/4 (B) 1/2 (C) 1 (D) 4 (E) 17

**8.19.** **[4-S01:3]** You are given the following times of first claim for five randomly selected auto insurance policies observed from time $t = 0$:

1 2 3 4 5

Calculate the kurtosis of this sample.

(A) 0.0 (B) 0.5 (C) 1.7 (D) 3.4 (E) 6.8

**8.20.** **[110-S92:4]** Let $X$ be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{30}x(1+3x) & \text{for } 1 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

Calculate $\mathbf{E}[1/X]$.

(A) $\frac{1}{12}$ (B) $\frac{7}{15}$ (C) $\frac{45}{103}$ (D) $\frac{11}{20}$ (E) $\frac{14}{15}$

**8.21.** A random variable $X$ has the probability density function

$$f(x) = \begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathbf{E}[1/X]$.



Exercises continue on the next page . . .

**8.22.** **[110-S83:12]** Let $X$ have the density function[1]

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

If $\beta = 6$ and $\alpha = 5$, what is the mean of $(1-X)^{-4}$?

(A) 42 (B) 63 (C) 210 (D) 252 (E) 315

**8.23.** **[Sample:129]** The proportion $X$ of yearly dental claims that exceed 200 is a random variable with probability density function

$$f(x) = \begin{cases} 60x^3(1-x)^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Calculate $\operatorname{Var}[X/(1-X)]$.

(A) 149/900 (B) 10/7 (C) 6 (D) 8 (E) 10

## Solutions

**8.1.** Adding a constant (such as 2.5) does not change the variance, and multiplying by a constant (like 1.03) multiplies the variance by the square of the constant. So the variance of the operating cost in the second year is $1.03^2(50) = \boxed{\mathbf{53.045}}$. **(B)**

**8.2.** It is not true that $\mathbf{E}[X^2] = \mathbf{E}[X]^2$, but expected value is linear.

$$\mathbf{E}[(X+5)^2] = \mathbf{E}[X^2 + 10X + 25] = \mathbf{E}[X^2] + 10\,\mathbf{E}[X] + 25$$

We can back out $\mathbf{E}[X^2]$ from $\mathbf{E}[X]$ and $\operatorname{Var}(X)$.

$$E[X^2] = \operatorname{Var}(X) + \mathbf{E}[X]^2 = 40 + 10^2 = 140$$

Finishing up,

$$\mathbf{E}[(X+5)^2] = 140 + 10(10) + 25 = \boxed{\mathbf{265}}$$

**8.3.** The means are the same since both graphs are centered at the same point. II varies more than I; there is more density in the tails of II than in I; thus the variance of II is greater than the variance of I. This leads to **(D)**

**8.4.** From $\mathbf{E}[(X-2)^3] = 0$, we expand $(X-2)^3$ and get

$$\mathbf{E}[X^3] - 6\,\mathbf{E}[X^2] + 12\,\mathbf{E}[X] - 8 = 0$$

Substituting for $\mathbf{E}[X^3]$ and $\mathbf{E}[X]$, we get

$$9 - 6\,\mathbf{E}[X^2] + 12(2) - 8 = 0$$

so $\mathbf{E}[X^2] = \frac{25}{6}$. Then $\operatorname{Var}(X) = \frac{25}{6} - 2^2 = \boxed{\frac{1}{6}}$. **(A)**

[1] The gamma function $\Gamma(x)$ is a continuous generalization of the factorial function. $\Gamma(n) = (n-1)!$ for $n$ a positive integer, and $\Gamma(x+1) = x\Gamma(x)$ for real $x$.

**8.5.** For any random variable $X$, $\mathrm{Var}(kX) = k^2\,\mathrm{Var}(X)$, so if $X$ is the annual cost of maintaining and repairing a car, then

$$\mathrm{Var}(1.2X) = 1.2^2\,\mathrm{Var}(X) = 1.44(260) = \boxed{\mathbf{374.4}} \qquad \textbf{(E)}$$

**8.6.** $F(1) = 1/2$, so we have a mixed distribution here. However, $F(2^-) = F(2) = 1$.

The density function between 1 and 2 is

$$f(x) = F'(x) = \frac{2x-2}{2} = x - 1$$

When calculating $k^{\text{th}}$ moments, we must add $1/2(1^k) = 1/2$ to $\int_1^2 x^k f(x)\,\mathrm{d}x$ to account for the point mass at 1. The moments of $X$ are

$$\begin{aligned}
\mathbf{E}[X] &= \frac{1}{2} + \int_1^2 x f(x)\,\mathrm{d}x \\
&= \frac{1}{2} + \int_1^2 (x^2 - x)\mathrm{d}x \\
&= \frac{1}{2} + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_1^2 \\
&= \frac{1}{2} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = \frac{4}{3} \\
\mathbf{E}[X^2] &= \frac{1}{2} + \int_1^2 x^2 f(x)\,\mathrm{d}x \\
&= \frac{1}{2} + \int_1^2 (x^3 - x^2)\mathrm{d}x \\
&= \frac{1}{2} + \left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Bigg|_1^2 \\
&= \frac{1}{2} + 4 - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} = \frac{23}{12}
\end{aligned}$$

The variance of $X$ is

$$\frac{23}{12} - \left(\frac{4}{3}\right)^2 = \boxed{\mathbf{\frac{5}{36}}} \qquad \textbf{(C)}$$

**8.7.** We calculate the mean and variance of claim size, $X$. This can be done easily on a statistical calculator.

$$\begin{aligned}
\mathbf{E}[X] &= 0.15(20) + 0.10(30) + 0.05(40) + 0.20(50) + 0.10(60 + 70) + 0.30(80) = 55 \\
\mathbf{E}[X^2] &= 0.15(20^2) + 0.10(30^2) + \cdots + 0.30(80^2) = 3500 \\
\mathrm{Var}(X) &= 3500 - 55^2 = 475
\end{aligned}$$

The standard deviation is $\sqrt{475} \approx 22$, and the probability that $33 < X < 77$ is $0.05 + 0.20 + 0.10 + 0.10 = \boxed{\mathbf{0.45}}$. **(A)**

**8.8.** The insurance policy pays 300 for [50,60), 600 for [60,70), and 700 for $[70,\infty)$. Let $X$ be the payment. The standard deviation of this variable can be calculated using a statistical calculator, but we'll work it out.

$$\begin{aligned}
\mathbf{E}[X] &= 0.14(300) + 0.06(600) + (0.04 + 0.04)(700) = 134 \\
\mathbf{E}[X^2] &= 0.14(300^2) + 0.06(600^2) + 0.08(700^2) = 73{,}400 \\
\sqrt{\mathrm{Var}(X)} &= \sqrt{73{,}400 - 134^2} = \boxed{\mathbf{235.47}} \qquad \textbf{(B)}
\end{aligned}$$

**8.9.** There are four categories of events: payments of 0, 200, 400, and 500. 0 is paid for less than 50 inches, 200 for 50–60 inches, 400 for 60–70 inches, and 500 for 70 or more inches. The probabilities of the four payments, based on the table, are:

| Payment | Probability |
|---|---|
| 0 | 0.72 |
| 200 | 0.14 |
| 400 | 0.06 |
| 500 | 0.08 |

We'll calculate first and second moments, then standard deviation.

$$\mathbf{E}[X] = 0.14(200) + 0.06(400) + 0.08(500) = 92$$
$$\mathbf{E}[X^2] = 0.14(200^2) + 0.06(400^2) + 0.08(500^2) = 35{,}200$$
$$\sqrt{E[X^2] - \mathbf{E}[X]^2} = \sqrt{35{,}000 - 92^2} = \boxed{163.5} \qquad \textbf{(A)}$$

**8.10.** Let's summarize the table based on distinct payments.

| Inches of Rain | Probability | Payment |
|---|---|---|
| $0 \le x < 14$ | 0.70 | 0 |
| $14 \le x < 16$ | 0.11 | 5000 |
| $16 \le x < 18$ | 0.08 | 10000 |
| $18 \le x < 20$ | 0.07 | 15000 |
| $20 \le x$ | 0.04 | 18000 |

Now let's calculate first and second moments. Let $X$ be the payment.

$$E[X] = 0.11(5000) + 0.08(10000) + 0.07(15000) + 0.04(18000) = 3120$$
$$E[X^2] = 0.11(5000^2) + 0.08(10000^2) + 0.07(15000^2) + 0.04(18000^2) = 39{,}460{,}000$$
$$\sqrt{\mathrm{Var}(X)} = \sqrt{39{,}460{,}000 - 3120^2} = \boxed{5452.119} \qquad \textbf{(D)}$$

**8.11.** Given that no spade is selected, there are 7 cards. Probability of 0 diamonds is $(5/7)(4/6) = 20/42$. Probability of 2 diamonds is $(2/7)(1/6) = 2/42$. Otherwise 1 diamond is selected, probability $1 - \frac{20}{42} - \frac{2}{42} = \frac{20}{42}$. We'll calculate first and second moments of number of diamonds $N$ and then the variance.

$$\mathbf{E}[N] = \frac{20(0) + 20(1) + 2(2)}{42} = \frac{24}{42}$$
$$\mathbf{E}[N^2] = \frac{20(0^2) + 20(1^2) + 2(2^2)}{42} = \frac{28}{42}$$
$$\mathrm{Var}(N) = \frac{28}{42} - \left(\frac{24}{42}\right)^2 = \boxed{0.340136} \qquad \textbf{(D)}$$

**8.12.** $\mathrm{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$. If we had $\mathbf{E}[X]$, we'd be done. To get $\mathbf{E}[x]$, expand $(X-1)^2$.

$$\mathbf{E}[(X-1)^2] = \mathbf{E}[X^2] - 2\,\mathbf{E}[X] + 1$$
$$47 = 61 - 2\,\mathbf{E}[X] + 1$$
$$\mathbf{E}[X] = 7.5$$
$$\sqrt{\mathrm{Var}(X)} = \sqrt{61 - 7.5^2} = \boxed{2.179} \qquad \textbf{(A)}$$

**8.13.** Mean and variance for percentage change of Stock X are

$$\begin{aligned}\mathbf{E}[\Delta_X] &= 0.3(0.18) + 0.5(0.08) + 0.2(-0.13) = 0.068\\ \mathbf{E}[\Delta_X^2] &= 0.3(0.18)^2 + 0.5(0.08)^2 + 0.2(-0.13)^2 = 0.0163\\ \operatorname{Var}(\Delta_X) &= 0.0163 - 0.068^2 = 0.011676\end{aligned}$$

Mean and variance for percentage change of Stock Y are

$$\begin{aligned}\mathbf{E}[\Delta_Y] &= 0.3(0.15) + 0.5(0.07) + 0.2(-0.06) = 0.068\\ \mathbf{E}[\Delta_Y^2] &= 0.3(0.15)^2 + 0.5(0.07)^2 + 0.2(-0.06)^2 = 0.00992\\ \operatorname{Var}(\Delta_Y) &= 0.0163 - 0.068^2 = 0.005296\end{aligned}$$

We see that **(B)** is true.

We could have saved time by not working out the exact variances. It's obvious that $Y$'s changes have a smaller range, so it will have a lower variance.

**8.14.** We calculate the first two raw moments and then use $\operatorname{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$.

$$\begin{aligned}\mathbf{E}[X] &= \int_{500}^{2000} \frac{500\,dx}{x} + 2000\bigl(1 - F(2000)\bigr)\\ 1 - F(2000) &= \int_{2000}^{\infty} \frac{500\,dx}{x^2} = -\frac{500}{x}\bigg|_{2000}^{\infty} = 0.25\\ \mathbf{E}[X] &= 500(\ln 2000 - \ln 500) + 0.25(2000) = 1193.15\\ \mathbf{E}[X^2] &= \int_{500}^{2000} 500\,dx + 2000^2\bigl(1 - F(2000)\bigr)\\ &= 500(1500) + 2000^2(0.25) = 1{,}750{,}000\\ \operatorname{Var}(X) &= 1{,}750{,}000 - 1193.15^2 = \boxed{\mathbf{326{,}400}}\end{aligned}$$

**8.15.** Since coefficient of variation is unitless, it is the same regardless of $c$. Between 0 and $c$:

$$f(x) = \frac{1}{c}$$

Let CV be the coefficient of variation.

$$\begin{aligned}\mathbf{E}[X] &= \int_0^c \frac{x\,dx}{c} = \frac{c}{2}\\ \mathbf{E}[X^2] &= \int_0^c \frac{x^2\,dx}{c} = \frac{c^2}{3}\\ \operatorname{Var}(X) &= \frac{c^2}{3} - \left(\frac{c}{2}\right)^2 = \frac{c^2}{12}\\ \mathrm{CV} &= \frac{c/\sqrt{12}}{c/2} = \frac{2}{\sqrt{12}} = \boxed{\mathbf{0.577350}}\end{aligned}$$

**8.16.** You can use your statistics calculator to calculate weighted means and variances. We'll work them out.

$$\begin{aligned}\mathbf{E}[N] &= 0.5 + 0.2(2) = 0.9\\ \mathbf{E}[N^2] &= 0.5 + 0.2(4) = 1.3\end{aligned}$$

$$\begin{aligned}\operatorname{Var}(N) &= 1.3 - 0.9^2 = 0.49\\ \mathbf{E}[(N-0.9)^3] &= 0.3(-0.9)^3 + 0.5(0.1)^3 + 0.2(1.1)^3 = 0.048\\ \text{Skewness} &= \frac{0.048}{0.49^{3/2}} = \boxed{0.139942}\end{aligned}$$

**8.17.** The distribution is symmetric, so its skewness is $\boxed{0}$. **(B)** If this is not obvious to you, calculate the mean, which is 300. Then note that

$$\mathbf{E}[X^3] = 0.05(100-300)^3 + 0.20(200-300)^3 + 0.20(400-300)^3 + 0.05(500-300)^3 = 0$$

so the coefficient of skewness, which is $\mu_3$ divided by $\sigma^3$, is 0.

**8.18.** From the coefficient of variation, we have

$$\begin{aligned}\frac{\sigma}{\mu} &= 2\\ \sigma &= 4\\ \sigma^2 &= 16\\ \mathbf{E}[X^2] &= \sigma^2 + \mu^2 = 16 + 2^2 = 20\\ \sigma^3 &= 64\\ \text{Skewness} &= \frac{\mathbf{E}[(X-\mu)^3]}{\sigma^3} = \frac{136 - 3(20)(2) + 2(8)}{64} = \boxed{\frac{1}{2}} \qquad \textbf{(B)}\end{aligned}$$

**8.19.** The variance is

$$\sigma^2 = \frac{(1-3)^2 + (2-3)^2 + (4-3)^2 + (5-3)^2}{5} = 2$$

The fourth central moment is

$$\mathbf{E}[(X-\mu)^4] = \frac{(1-3)^4 + (2-3)^4 + (4-3)^4 + (5-3)^4}{5} = 6.8$$

Kurtosis is

$$\text{Kurtosis} = \frac{\mathbf{E}[(X-\mu)^4]}{\sigma^4} = \frac{6.8}{2^2} = \boxed{1.7} \qquad \textbf{(C)}$$

**8.20.** Since $f(x)$ is nonzero only on $[1,3]$, we integrate from 1 to 3.

$$\begin{aligned}\mathbf{E}\left[\frac{1}{X}\right] &= \int_1^3 \frac{1}{x} f(x)\,dx\\ &= \frac{1}{30}\int_1^3 (1+3x)dx\\ &= \frac{1}{30}(x+1.5x^2)\Big|_1^3\\ &= \frac{16.5-2.5}{30} = \boxed{\frac{7}{15}} \qquad \textbf{(B)}\end{aligned}$$

**8.21.**

$$\begin{aligned}\mathbf{E}\left[\frac{1}{X}\right] &= \int_0^1 6(1-x)\,dx\\ &= -6\left(\frac{(1-x)^2}{2}\right)\Big|_0^1\\ &= \boxed{3}\end{aligned}$$

**8.22.** We integrate $(1-x)^{-4}$ over the density function.

$$\int_0^1 \frac{\Gamma(11)}{\Gamma(5)\Gamma(6)} x^4(1-x)^5(1-x)^{-4}\mathrm{d}x = \int_0^1 \frac{10!}{4!5!}(x^4 - x^5)\,\mathrm{d}x$$

$$= 1260\left(\frac{x^5}{5} - \frac{x^6}{6}\right)\Bigg|_0^1 = 1260\left(\frac{1}{5} - \frac{1}{6}\right) = \boxed{42} \qquad \textbf{(A)}$$

**8.23.** Calculate the first and second moments.

$$\mathbf{E}\left[\frac{X}{1-X}\right] = \int_0^1 60\left(\frac{x}{1-x}\right)x^3(1-x)^2\,\mathrm{d}x = 60\int_0^1 x^4(1-x)\mathrm{d}x = 60\left(\frac{x^5}{5} - \frac{x^6}{6}\right)\Bigg|_0^1 = 2$$

$$\mathbf{E}\left[\left(\frac{X}{1-X}\right)^2\right] = \int_0^1 60\left(\frac{x^2}{(1-x)^2}\right)x^3(1-x)^2\,\mathrm{d}x = 60\int_0^1 x^5\,\mathrm{d}x = 60\left(\frac{x^6}{6}\right)\Bigg|_0^1 = 10$$

$$\mathrm{Var}\left(\frac{X}{1-X}\right) = 10 - 2^2 = \boxed{6} \qquad \textbf{(C)}$$

# Lesson 15

# Uniform Distribution

The uniform distribution is a standard distribution. We will discuss many standard distributions in later lesson, but we discuss the uniform distribution now because it is used in examples in the next lesson.

## Continuous Uniform

A continuous uniform distribution on interval $[a, b]$ has constant density on the interval $[a, b]$ and zero density elsewhere. The probability density function must integrate to 1, so the constant density must be the reciprocal of the length of the interval, $1/(b-a)$. Typically, the interval starts at 0. A typical uniform distribution is defined by

$$f(x) = \begin{cases} \dfrac{1}{\theta} & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{x}{\theta} & 0 < x \le \theta \\ 1 & x > \theta \end{cases}$$

The mean of a uniform distribution is the midpoint of the interval. For a uniform random variable $X$ on $[0, \theta]$, $\mathbf{E}[X] = \theta/2$.

The variance of a uniform distribution is the interval length squared divided by 12. For a uniform random variable $X$ on $[0, \theta]$, $\operatorname{Var}(X) = \theta^2/12$.

The median of a uniform distribution is the midpoint of the range; the midpoint is the mean. The $100p^{\text{th}}$ percentile of a uniform distribution on the interval $[a, b]$ is $(1-p)a + pb$.

Suppose $a < b < c$. Then if $X$ is uniform on $[a, c]$, then the conditional distribution of $X$ given $X > b$ is uniform on $[b, c]$. And the conditional distribution of $X$ given $X < b$ is uniform on $[a, b]$.

**Example 15A** Losses on an insurance coverage are uniformly distributed on $(0, 1000]$. The insurance policy has a policy limit of 600.

Calculate the average payment made by the policy for a loss.

**Solution:** The density function of the uniform distribution is $1/1000$ for $0 < x \le 1000$, 0 otherwise. Since the policy limit is 500, we integrate $x$ times the density function up to 600. We add to that 500 times the probability that the loss is greater than 600. Let $X$ be the loss and let $Y$ be the payment per loss.

$$\mathbf{E}[Y] = \int_0^{600} \frac{y\,dy}{1000} + 500\Pr(X > 600)$$

$$\Pr(X > 600) = 1 - \frac{600}{1000} = 0.4$$

$$\mathbf{E}[Y] = \left.\frac{y^2}{2000}\right|_0^{600} + 600(0.4) = \frac{600^2}{2000} + 200 = 180 + 240 = \boxed{420}$$

An easier way to work this out is to use the double expectation formula. Condition on $X$ being below or above 600.

$$\mathbf{E}[Y] = \Pr(X \le 600)\,\mathbf{E}[X \mid X \le 600] + 600\Pr(X > 600)$$

Now, the conditional variable $X \mid X \le 600$ is uniform, so its mean is 300. We have

$$\mathbf{E}[Y] = 0.6(300) + 0.4(600) = 180 + 240 = \boxed{420}$$

□

## Discrete Uniform

Sometimes we may speak about a discrete uniform distribution. Such a distribution has equal probabilities on all integers in a range. For example, a discrete uniform distribution may assign probabilities of 1/6 to each integer from 1 to 6. However, when we specify a uniform distribution and don't specify that it is discrete, it is a continuous uniform distribution.

The moments of a discrete uniform distribution $N$ on {1,. . . ,n} are

$$\mathbf{E}[N] = \frac{n+1}{2} \tag{15.1}$$

$$\mathrm{Var}(N) = \frac{n^2-1}{12} \tag{15.2}$$

I doubt you'll be expected to know the variance formula on an exam.

**Example 15B** The number of claims on policy A is uniformly distributed on $\{0,1,2,3\}$. The number of claims on policy B is uniformly distributed on $\{0,1,2\}$. The two policies are independent.

Calculate the probability that policy A makes more claims than policy B.

**Solution:** The claim counts on policy A each have probability 1/4 and the claim counts on policy B each have probability 1/3, so the joint probabilities are 1/12. Let $(x,y)$ be a pair in which $x$ is the number of claims on policy A and $y$ is the number of claims on policy B. The pairs with more claims for policy A are (1,0), (2,0), (2,1), and all three pairs starting with 3, for a total of six pairs. The probability of more claims on policy A is $6/12 = \boxed{0.5}$. □

## Beta Distribution

A generalization of the continuous uniform distribution is the *beta distribution*. This distribution has two parameters, $a$ and $b$. Its density function is

$$f(x;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad 0 \le x \le 1$$

Recall that $\Gamma(x+1) = x\Gamma(x)$. so the ratio of Gammas in this formula is easy to compute; it equals $(a+b-1)(a+b-2)\cdots(a)$. The uniform distribution on $[0,1]$ is a special case with $a = b = 1$. It is possible to scale a beta distribution so that it has nonzero density on $[0,\theta]$.

Let $X$ be a beta random variable with parameters $a$ and $b$. The moments of $X$ are

$$\mathbf{E}[X] = \frac{a}{a+b} \tag{15.3}$$

$$\mathrm{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)} \tag{15.4}$$

I doubt that you will need the variance formula for an exam.

The mode is $(a-1)/(a+b-2)$, assuming the denominator is positive.

The beta distribution may be used for random variables that must be between 0 and 1. For example, if the parameter $p$ of a binomial distribution is unknown it may be modeled with a beta distribution.

**Table 15.1:** Summary of uniform distribution

### Continuous uniform on $[0, \theta]$

$$f(x) = \frac{1}{\theta} \qquad 0 \le x \le \theta$$
$$F(x) = \frac{x}{\theta} \qquad 0 \le x \le \theta$$
$$\mathbf{E}[X] = \frac{\theta}{2}$$
$$\mathrm{Var}(X) = \frac{\theta^2}{12}$$

### Discrete Uniform on 1, 2, . . . , $n$

$$f(x) = \frac{1}{n} \qquad x = 1, 2, \ldots, n$$
$$\mathbf{E}[X] = \frac{n+1}{2} \tag{15.1}$$
$$\mathrm{Var}(X) = \frac{n^2-1}{12} \tag{15.2}$$

### Beta Distribution

$$\mathbf{E}[X] = \frac{a}{a+b} \tag{15.3}$$
$$\mathrm{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)} \tag{15.4}$$
$$\text{Mode} = \frac{a-1}{a+b-2} \quad \text{when } a+b-2 > 0 \tag{15.5}$$

# Exercises

## Continuous Uniform

### Cumulative distribution function

**15.1.** An insurer offers a travelers insurance policy. Losses under the policy are uniformly distributed on the interval $[0, 10]$.

The insurer reimburses a policyholder for a loss subject to a deductible of 2.

Determine the cumulative distribution function, $F$, of the benefit that the insurer pays a policyholder who experiences exactly one loss under the policy.

**15.2.** **[Sample:131]** An insurer offers a travelers insurance policy. Losses under the policy are uniformly distributed on the interval $[0, 5]$.

The insurer reimburses a policyholder for a loss up to a maximum of 4.

Determine the cumulative distribution function, $F$, of the benefit that the insurer pays a policyholder who experiences exactly one loss under the policy.

(A) $F(x) = \begin{cases} 0, & x < 0 \\ 0.20x, & 0 \le x < 4 \\ 1, & x \ge 4 \end{cases}$

(B) $F(x) = \begin{cases} 0, & x < 0 \\ 0.20x, & 0 \le x < 5 \\ 1, & x \ge 5 \end{cases}$

(C) $F(x) = \begin{cases} 0, & x < 0 \\ 0.25x, & 0 \le x < 4 \\ 1, & x \ge 4 \end{cases}$

(D) $F(x) = \begin{cases} 0, & x < 0 \\ 0.25x, & 0 \le x < 5 \\ 1, & x \ge 4 \end{cases}$

(E) $F(x) = \begin{cases} 0, & x < 1 \\ 0.25x, & 1 \le x < 5 \\ 1, & x \ge 5 \end{cases}$

### Probability

**15.3.** **[Sample:141]** Losses covered by a flood insurance policy are uniformly distributed on the interval $[0, 2]$. The insurer pays the amount of the loss in excess of a deductible $d$.

The probability that the insurer pays at least 1.20 on a random loss is 0.30.

Calculate the probability that the insurer pays at least 1.44 on a random loss.

(A) 0.06 (B) 0.16 (C) 0.18 (D) 0.20 (E) 0.28

**15.4.** **[Sample:348]** A policyholder incurs one loss under each of three policies. Each policy has a deductible of 30. Losses under each policy are uniformly distributed on the interval [0, 100].

The three losses are mutually independent.

Calculate the probability that the policyholder will receive benefits from any of the three policies.

(A) 0.027 (B) 0.343 (C) 0.657 (D) 0.700 (E) 0.973



*Exercises continue on the next page . . .*

**15.5.** [**Sample:448**] A policyholder sustains one loss covered by policy A and a second loss covered by policy B. The two losses are independent and uniformly distributed on the interval $[0, 10]$. Each policy has a deductible of 5.

Calculate the probability that the larger of the two claim payments does not exceed $t$, for $0 \le t \le 5$.

(A) $\left(\frac{t}{5}\right)^2$ (B) $\left(\frac{t}{10}\right)^2$ (C) $\frac{5+t}{10}$ (D) $\left(\frac{5+t}{10}\right)^2$ (E) $1-\left(\frac{5-t}{10}\right)^2$

**15.6.** [**Sample:108**] In a certain game of chance, a square board with area 1 is colored with sectors of either red or blue. A player, who cannot see the board, must specify a point on the board by giving an $x$-coordinate and a $y$-coordinate. The player wins the game if the specified point is in a blue sector. The game can be arranged with any number of red sectors, and the red sectors are designed so that

$$R_i = \left(\frac{9}{20}\right)^i, \text{ where } R_i \text{ is the area of the } i^{\text{th}} \text{ sector}$$

Calculate the minimum number of red sectors that makes the chance of a player winning less than 20%.

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

**15.7.** [**Sample:454**] Losses under an insurance policy are uniformly distributed on $[0, 1000]$. The policy has a deductible of 400.

A loss occurred for which the insurance benefit was less than 400.

Calculate the probability that the benefit was more than 300.

(A) 0.100 (B) 0.125 (C) 0.250 (D) 0.750 (E) 0.875

**15.8.** [**Sample:123**] A policyholder has probability 0.7 of having no claims, 0.2 of having exactly one claim, and 0.1 of having exactly two claims. Claim amounts are uniformly distributed on the interval [0, 60] and are independent. The insurer covers 100% of each claim.

Calculate the probability that the total benefit paid to the policyholder is 48 or less.

(A) 0.320 (B) 0.400 (C) 0.800 (D) 0.892 (E) 0.924

**15.9.** [**Sample:285**] Appraisals of the value of a necklace are uniformly distributed on the interval $[\theta - 3, \theta + 1]$, where $\theta$ is the actual price the owner paid for the necklace. Four mutually independent appraisals are obtained.

Let $L$ denote the lowest of the four appraisals and $H$ the highest.

Calculate $P[L < \theta < H]$.

(A) 0.152 (B) 0.188 (C) 0.600 (D) 0.680 (E) 0.996

**15.10.** [**Sample:447**] The amount of money stolen from an insured home during a burglary is modeled by a random variable that is uniformly distributed on the interval $[0, 1000]$. The claim payment that the insurer makes for such a loss under its homeowners policy has the following characteristics:

i) The claim payment equals a constant percentage, $p$, of the amount by which the loss exceeds 400.
ii) The expected claim payment is 90.

Calculate $p$.

(A) 15% (B) 18% (C) 30% (D) 50% (E) 75%



*Exercises continue on the next page . . .*

**15.11.** **[Sample:237]** A car and a bus arrive at a railroad crossing at times independently and uniformly distributed between 7:15 and 7:30. A train arrives at the crossing at 7:20 and halts traffic at the crossing for five minutes.

Calculate the probability that the waiting time of the car or the bus at the crossing exceeds three minutes.

(A) 0.25 (B) 0.27 (C) 0.36 (D) 0.40 (E) 0.56

### Expected value

**15.12.** **[110-S83:13]** A box is to be constructed so that its height is 10 inches and its base is $X$ inches by $X$ inches. If $X$ has a uniform distribution over the interval (2,8), then what is the expected volume of the box in cubic inches?

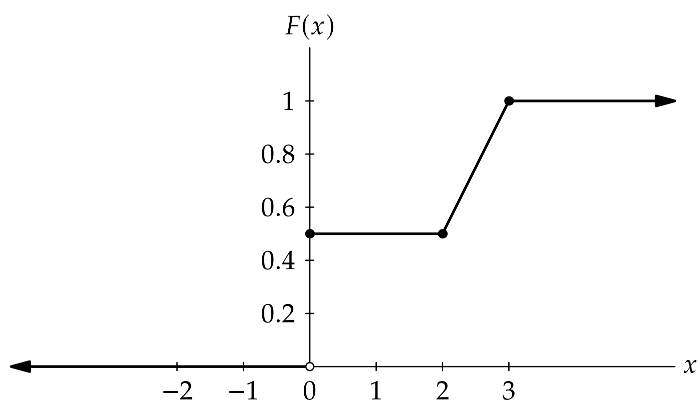
(A) 80.0 (B) 250.0 (C) 252.5 (D) 255.0 (E) 280.0

**15.13.** A side of a square is measured with a ruler. The error of the measurement is uniformly distributed on $[-1/8, 1/8]$. The area of the square is then estimated as the square of the measurement.

The true length of the side of the square is 10.

Calculate the expected value of the estimate of the area.

**15.14.** **[Sample:112]** (See also exercise 7.19, page 115) The figure below shows the cumulative distribution function of a random variable, $X$.




Calculate $\mathbf{E}[X]$.

(A) 0.00 (B) 0.05 (C) 1.00 (D) 1.25 (E) 2.50

**15.15.** **[S00:38,Sample:53]** An insurance policy is written to cover a loss, $X$, where $X$ has a uniform distribution on $[0, 1000]$. The policy has a deductible, $d$, and the expected payment under the policy is 25% of what it would be with no deductible.

Calculate $d$.

(A) 250 (B) 375 (C) 500 (D) 625 (E) 750



Exercises continue on the next page . . .

## Percentile

**15.16.** **[Sample:220]** A motorist just had an accident. The accident is minor with probability 0.75 and is otherwise major. Let $b$ be a positive constant. If the accident is minor, then the loss amount follows a uniform distribution on the interval $[0, b]$. If the accident is major, then the loss amount follows a uniform distribution on the interval $[b, 3b]$.

The median loss amount due to this accident is 672.

Calculate the mean loss amount due to this accident.

(A) 392 (B) 512 (C) 672 (D) 882 (E) 1008

**15.17.** **[Sample:181]** Losses covered by an insurance policy are modeled by a uniform distribution on the interval $[0, 1000]$. An insurance company reimburses losses in excess of a deductible of 250.

Calculate the difference between the median and the $20^{\text{th}}$ percentile of the insurance company reimbursement, over all losses.

(A) 225 (B) 250 (C) 300 (D) 375 (E) 500

**15.18.** **[Sample:257]** Let $X$ be the percentage score on a college-entrance exam for students who did not participate in an exam-preparation seminar. $X$ is modeled by a uniform distribution on $[a, 100]$.

Let $Y$ be the percentage score on a college-entrance exam for students who did participate in an exam-preparation seminar. $Y$ is modeled by a uniform distribution on $[1.25a, 100]$.

It is given $\mathbf{E}[X^2] = \dfrac{19{,}600}{3}$.

Calculate the $80^{\text{th}}$ percentile of $Y$.

(A) 80 (B) 85 (C) 90 (D) 92 (E) 95

**15.19.** **[Sample:484]** An insurance company has customer service operations in Denver, Philadelphia, and Salt Lake City.

Employee salaries in Denver are uniformly distributed from 25 to 90. Employee salaries in Philadelphia are uniformly distributed from 45 to $x$. Employee salaries in Salt Lake City are uniformly distributed from 10 to $x/3$.

The $40^{\text{th}}$ percentile of Denver salaries is equal to the $20^{\text{th}}$ percentile of Philadelphia salaries.

Calculate the median of Salt Lake City employee salaries.

(A) 12.5 (B) 17.5 (C) 25.0 (D) 35.0 (E) 60.0

**15.20.** **[Sample:485]** The loss due to a warehouse robbery is modeled by a uniform distribution on the interval $[a, 2a]$, where $a$ is a positive constant.

The ratio of the $40^{\text{th}}$ percentile of the loss to the $p^{\text{th}}$ percentile of the loss equals the ratio of the $p^{\text{th}}$ percentile of the loss to the $80^{\text{th}}$ percentile of the loss.

Calculate $p$.

(A) 56.6 (B) 58.7 (C) 60.0 (D) 61.4 (E) 65.4

## Variance

**15.21.** **[110-S88:40]** Let $X$ be a random variable with a uniform distribution on the interval $(1, a)$ where $a > 1$.

If $\mathbf{E}[X] = 6\operatorname{Var}(X)$, then $a =$

(A) 2 (B) 3 (C) $3\sqrt{2}$ (D) 7 (E) 8



*Exercises continue on the next page . . .*

**15.22.** **[Sample:479]** A patient must undergo hospitalization and surgery. The hospitalization and surgery charges are modeled by random variables uniformly distributed on the intervals $[0, c]$ and $[0, 3c - 18]$, respectively, where $c$ is a constant larger than 6.

The standard deviation of the hospitalization charge is $4\sqrt{3}$.

Calculate the standard deviation of the surgery charge.

(A) 2.8 (B) 7.3 (C) 10.4 (D) 15.6 (E) 20.8

**15.23.** **[Sample:441]** A patient must undergo hospitalization and surgery. The hospitalization and surgery charges are uniformly distributed on the intervals $[0, b]$ and $[0, 2b - 6]$, respectively, where $b$ is a constant larger than 3.

The standard deviation of the hospitalization charge is 9.60.

Calculate the standard deviation of the surgery charge.

(A) 13.2 (B) 15.7 (C) 17.5 (D) 19.2 (E) 19.9

**15.24.** **[Sample:442]** Let $X$ be a random variable that is uniform on $[a, b]$. The probability that $X$ is greater than 8 is 0.60. The probability that $X$ is greater than 11 is 0.20.

Calculate the variance of $X$.

(A) 3.70 (B) 4.69 (C) 6.25 (D) 7.24 (E) 8.75

**15.25.** **[Sample:453]** Losses under a policy are uniformly distributed on the interval $[0, 480]$. For each loss, the claim payment is a constant percentage of the amount in excess of a deductible of 240.

The insurer wants the variance of the claim payment for a single loss to equal 2000.

Calculate the percentage the insurer should choose.

(A) 11.1% (B) 33.3% (C) 57.7% (D) 64.5% (E) 91.3%

**15.26.** **[Sample:464]** The loss due to an injury in a certain sport is uniformly distributed on an interval.

The interquartile range of a random variable is defined as the difference between its $75^{\text{th}}$ and $25^{\text{th}}$ percentiles.

Determine the correct statement about the ratio of the standard deviation to the interquartile range of the loss due to a given injury in that sport.

(A) The ratio is $1 : \sqrt{3}$, regardless of the endpoints of the interval.
(B) The ratio is $1 : 1$, regardless of the endpoints of the interval.
(C) The ratio is $2 : \sqrt{3}$, regardless of the endpoints of the interval.
(D) The ratio depends on the length of the inteval.
(E) The ratio depends on the location of the center of the interval.

**15.27.** **[Sample:469]** Random variable $X$ follows a uniform distribution with mean 12 and $75^{\text{th}}$ percentile 18.

Calculate $\text{Var}(X)$.

(A) 24 (B) 36 (C) 48 (D) 144 (E) 192

**15.28.** **[Sample:473]** Losses under a boat insurance policy are uniformly distributed on the interval $[0, 1]$. The policy has a fixed deductible.

The expected value of the claim payment on a given loss is 0.245.

Calculate the variance of the claim payment on a given loss.

(A) 0.020 (B) 0.054 (C) 0.062 (D) 0.083 (E) 0.114



*Exercises continue on the next page . . .*

**15.29.** **[Sample:482]** A flight is delayed due to bad weather. The delay time is modeled by a random variable with a continuous uniform distribution. The expected delay time is three hours, and the standard deviation of the delay time is one hour.

Calculate the shortest possible delay time, in hours.

(A) 0.58 (B) 1.27 (C) 1.73 (D) 2.31 (E) 2.42

## Discrete uniform

**15.30.** **[Sample:150]** A theme park conducts a study of families that visit the park during a year. The fraction of such families of size $m$ is $\frac{8-m}{28}$, $m = 1, 2, 3, 4, 5, 6$, and 7.

For a family of size $m$ that visits the park, the number of members of the family that ride the roller coaster follows a discrete uniform distribution on the set $\{1, \ldots, m\}$.

Calculate the probability that a family visiting the park has exactly six members, given that exactly five members of the family ride the roller coaster.

(A) 0.17 (B) 0.21 (C) 0.24 (D) 0.28 (E) 0.31

**15.31.** **[Sample:170]** At a polling booth, ballots are cast by ten voters, of whom three are Republicans, two are Democrats, and five are Independents. A local journalist interviews two of these voters, chosen randomly.

Calculate the expectation of the absolute value of the difference between the number of Republicans interviewed and the number of Democrats interviewed.

(A) 1/5 (B) 7/15 (C) 3/5 (D) 11/15 (E) 1

**15.32.** **[Sample:109]** Automobile claim amounts are modeled by a uniform distribution on the interval $[0, 10{,}000]$. Actuary A reports $X$, the claim amount divided by 1000. Actuary B reports $Y$, which is $X$ rounded to the nearest integer from 0 to 10.

Calculate the absolute value of the difference between the $4^{\text{th}}$ moment of $X$ and the $4^{\text{th}}$ moment of $Y$.

(A) 0 (B) 33 (C) 296 (D) 303 (E) 533

**15.33.** **[Sample:481]** The number of days required for a damage control team to locate and repair a leak in the hull of a ship is modeled by a discrete random variable, $N$. $N$ is uniformly distributed on $\{1, 2, 3, 4, 5\}$.

The cost of locating and repairing a leak is $N^2 + N + 1$.

Calculate the expected cost of locating and repairing a leak in the hull of the ship.

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

# Solutions

**15.1.** There is a 0.2 probability that nothing is paid since the loss is below the deductible. If the loss is in $[2, 10]$, the payment is uniformly distributed on $[0, 8]$, so its conditional density (conditional on the loss being above 2) is $1/8$. Its unconditional density in $[2, 10]$ is $0.8(1/8) = 0.1$. The following cumulative distribution function results:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2 + 0.1x & 0 \le x < 8 \\ 1 & x \ge 8 \end{cases}$$

**15.2.** For a uniform distribution on [0,5], the cumulative distribution function $F^{\text{uniform}}$ is $F^{\text{uniform}}(x) = 0.2x$ for $0 \le x \le 5$. Our $F$ matches a uniform for $x < 4$. However, because of the policy limit of 4, the probability that the benefit is no greater than 4 is 1. In other words, $F(4) = 1$ for the $F$ representing the cumulative distribution function of the benefit. So **(A)** is the correct answer.

**15.3.** The probability that the loss is greater than $a$ for $0 \le a \le 2$ is $1 - a/2$. If this probability is 0.3, than $a = 1.4$. The insurer pays 1.20, so $d = 0.2$. The probability that the insurer pays at least 1.44 is the probability that the loss is greater than 1.64, or $1 - 1.64/2 = \boxed{0.18}$. **(C)**

**15.4.** The probability of not receiving benefits from one policy is $30/100 = 0.3$. The probability of not receiving benefits from 3 independent policies is $0.3^3 = 0.027$. The probability of receiving benefits from any policy is $1 - 0.027 = \boxed{0.973}$. **(E)**

**15.5.** We want the probability that both claim payments don't exceed $t$, and that is the square of the probability that one claim payment does not exceed $t$. That probability is the loss amount, $5 + t$, divided by 10. **(D)** is the answer.

**15.6.** The selected point is uniformly distributed on the square, so the probability of selecting a point in a blue sector is the total area of the blue sectors, which is 1 minus the total area of the red sectors. Let $n$ be the number of red sectors. The total area of the blue sectors is

$$1 - \sum_{i=1}^{n} \left(\frac{9}{20}\right)^i = 1 - \frac{\frac{9}{20} - \left(\frac{9}{20}\right)^{n+1}}{1 - \frac{9}{20}}$$

We want this to be less than 20%.

$$\begin{aligned}
1 - \frac{\frac{9}{20} - \left(\frac{9}{20}\right)^{n+1}}{1 - \frac{9}{20}} &\le 0.2 \\
\frac{\frac{9}{20} - \left(\frac{9}{20}\right)^{n+1}}{1 - \frac{9}{20}} &\ge 0.8 \\
\frac{9}{20} - \left(\frac{9}{20}\right)^{n+1} &\ge \frac{8.8}{20} = 0.44 \\
0.45^{n+1} &\le 0.01 \\
(n+1)\ln 0.45 &\le \ln 0.01 \\
n &\ge \frac{\ln 0.01}{\ln 0.45} - 1 = 4.767
\end{aligned}$$

To make the expression less than 0.2, $n$ must be at least $\boxed{5}$. **(C)**

**15.7.** Let $L$ be the loss. The benefit is less than 400 if the loss is less than 800, and the benefit is greater than 300 if the loss is greater than 700. We want

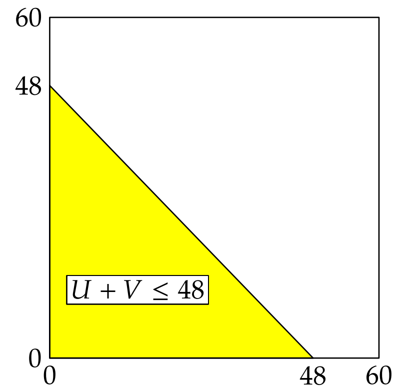
$$P[L < 800 \mid L > 700] = \frac{P[700 < L < 800]}{P[L < 800]} = \frac{1/10}{8/10} = \boxed{0.125} \qquad \textbf{(B)}$$

**15.8.** Use the Law of Total Probability, conditioning on the number of claims. If there are no claims, the total benefit is certainly 48 or less. If there is one claim, by uniformity, the probability of 48 or less is $48/60 = 0.8$. If there are 2 claims, the probability that the sum is 48 or less is the probability that the first claim $U$ is less than 48 and the second claim $V$ is less than $48 - U$. Let's calculate that probability. Note that the probability density function for each claim is $1/60$.

$$\begin{aligned}
\Pr(U + V \le 48) &= \int_0^{48} \Pr(V \le 48 - u) f(u)\, du \\
&= \int_0^{48} \left(\frac{48 - u}{60}\right)\left(\frac{1}{60}\right) du \\
&= \frac{1}{3600}\left(\frac{-(48-u)^2}{2}\right)\Bigg|_0^{48} = \frac{48^2}{7200} = 0.32
\end{aligned}$$

Perhaps a better way to derive this probability is geometrically. The joint distribution of $U$ and $V$ is uniform on the $60 \times 60$ square, and the sum is less than 48 on the triangle with vertices (0,0), (48,0), (0,48), as shown in the figure to the right. The square has area 3600 and the triangle has area $48^2/2$, leading to a probability of $48^2/7200 = 0.32$. So the probability that the total benefit is 48 or less is

$$0.7(1) + 0.2(0.8) + 0.1(0.32) = \boxed{0.892} \qquad \textbf{(D)}$$




**15.9.** The probability that an appraisal is less than $\theta$ is $3/4$, and the probability that an appraisal is greater than $\theta$ is $1/4$. The probability that at least one out of four such appraisals is less than $\theta$ is the complement of the probability that all four are greater than $\theta$, or $1 - (1/4)^4 = 0.996094$. So $P[L < \theta] = 0.996094$.

Similarly, $P[H > \theta] = 1 - (3/4)^4 = 0.683594$. Therefore

$$P[L < \theta < H] = (0.996094)(0.683594) = \boxed{0.680923} \qquad \textbf{(D)}$$

**15.10.** First we calculate the expected excess, per loss, of the loss over 400. Then, dividing 90 by that expected excess, we obtain $p$.

Let $L$ be the loss and $P$ be the excess. By the double expectation formula, the expected excess is

$$\mathbf{E}[P] = P[L \leq 400]\,\mathbf{E}[P \mid L \leq 400] + P[L > 400]\,\mathbf{E}[P \mid L > 400]$$

When the loss is greater than 400, the excess is uniform on $[0, 600]$ so the mean is 300.

$$\mathbf{E}[P] = 0.4(0) + 0.6(300) = 180$$

We conclude that $p = 90/180 = \boxed{0.5}$. **(D)**

**15.11.** Waiting time will exceed 3 minutes if a vehicle arrives between 7:20 and 7:22. The probability of a vehicle arriving in that 2-minute interval is $2/15$. Use the inclusion-exclusion principle to calculate the probability of at least one vehicle being delayed. The probability both are delayed is $\left(\frac{2}{15}\right)^2$.

$$\frac{2}{15} + \frac{2}{15} - \left(\frac{2}{15}\right)^2 = \boxed{0.24889} \qquad \textbf{(A)}$$

Alternatively, calculate the probability that neither is delayed, then take the complement:

$$1 - \left(\frac{13}{15}\right)^2 = \boxed{0.24889}$$

**15.12.** The volume of the box is $10X^2$. The expected value of the volume, since the probability density function of the uniform is $f(x) = 1/6$ between 2 and 8:

$$\mathbf{E}[10X^2] = 10\,\mathbf{E}[X^2] = 10\int_2^8 \frac{1}{6}x^2\,dx = \frac{10}{6}\,\frac{8^3 - 2^3}{3} = \boxed{280} \qquad \textbf{(E)}$$

**15.13.** The measurement $X$ is a uniform random variable on $9\frac{7}{8}, 10\frac{1}{8}$. Then $\mathbf{E}[X^2] = \mathrm{Var}(X) + \mathbf{E}[X]^2$, and $\mathbf{E}[X] = 10$, $\mathrm{Var}(X) = \frac{(1/4)^2}{12} = 1/192$, so we get

$$\mathbf{E}[X^2] = \frac{1}{192} + 10^2 = \boxed{100\tfrac{1}{192}}$$

**15.14.** From the diagram of $F(x)$, we see that $X$ has a point mass with probability 0.5 at 0 and is uniform on $[2, 3]$ with probability 0.5. Thus its expected value is $1/2$ of the expected value of a uniform on $[2, 3]$, or $0.5(2.5) = \boxed{1.25}$. **(D)**

**15.15.** With no deductible, the expected payment for a uniform distribution is the midpoint of the interval, or 500.

With a deductible $d$, the expected payment is the probability that the loss is above the deductible, $1 - d/1000$, times the midpoint of the payment after the deductible, $(1000 - d)/2$. So we want

$$\left(1 - \frac{d}{1000}\right)\left(\frac{1000-d}{2}\right) = 0.25(500) = 125$$
$$\left(\frac{1000-d}{1000}\right)\left(\frac{1000-d}{2}\right) = 125$$
$$(1000-d)^2 = 125(2)(1000) = 250{,}000$$
$$1000 - d = 500$$
$$d = \boxed{500} \qquad \textbf{(C)}$$

**15.16.** Let $X \ge 0$ be the loss amount. $\Pr(X \le b) = 0.75$, and the median is where $\Pr(X \le b) = 0.5$. Due to uniformity, assuming $0 \le x \le b$,

$$\Pr(X \le x) = 0.75 \Pr(X \le x \mid X \le b) = 0.75\left(\frac{x}{b}\right)$$

and setting this equal to 0.5, we get $x = 2b/3$. That $x$ is the median, and it equals 672, so $b = (3/2)(672) = 1008$. The mean loss amount, by the double expectation formula and using that the mean of a uniform is the midpoint, is

$$\mathbf{E}[X] = \Pr(X \le b)\,\mathbf{E}[X \mid X \le b] + \Pr(X > b)\,\mathbf{E}[X \mid X > b] = 0.75(504) + 0.25(2016) = \boxed{882} \qquad \textbf{(D)}$$

**15.17.** The median and $20^{\text{th}}$ percentiles of the payment are functions of the corresponding percentiles of the loss. For a uniform distribution on $[0, a]$, each percentile is the percent times the upper bound $a$, so the median loss is 500 and the $20^{\text{th}}$ percentile of loss is 200. Payments for these losses are 250 and 0 respectively, so the difference between them is $\boxed{250}$. **(B)**

**15.18.** The density of $X$ is the constant $1/(100 - a)$. The raw second moment of a uniform distribution on $[a, 100]$ is

$$\begin{aligned}\mathbf{E}[X^2] &= \int_a^{100} \frac{x^2\,dx}{100-a} \\ &= \frac{100^3 - a^3}{3(100-a)} \\ &= \frac{100^2 + 100a + a^2}{3}\end{aligned}$$

Setting this equal to 19,600/3,

$$a^2 + 100a - 9{,}600 = 0$$
$$a = \frac{-100 + \sqrt{48{,}400}}{2} = 60$$

Then $1.25a = 75$. For $Y$,

$$F_Y(x) = \frac{x - 75}{25} \qquad 75 \le x \le 100$$

and we want the $80^{\text{th}}$ percentile of $Y$, the $x$ for which $F_Y(x) = 0.8$, so $(1/25)(x - 75) = 0.8$ and we conclude that the $80^{\text{th}}$ percentile is $\boxed{95}$. **(E)**

**15.19.** The $40^{\text{th}}$ percentile of Denver salaries is obtained by linear interpolation between minimum and maximum: $0.6(25) + 0.4(90) = 51$. Since 51 is the $20^{\text{th}}$ percentile of Philadelphia salaries, and the minimum is 45. the maximum is such that $0.8(45) + 0.2x = 51$, so $x = 75$. Then $x/3 = 25$, so the median of Salt Lake City salaries is the midpoint of $[10, 25]$, or $\boxed{17.5}$. **(B)**

**15.20.** The 40th percentile of the loss is $a + 0.4(2a - a) = 1.4a$ and the 80th percentile is $1.8a$. The $p^{\text{th}}$ percentile of the loss is $(1 + 0.01p)a$. So

$$\begin{aligned}\frac{1.4a}{(1+0.01p)a} &= \frac{(1+0.01p)a}{1.8a}\\(1+0.01p)^2 &= (1.4)(1.8) = 2.52\\0.01p &= \sqrt{2.52} - 1 = 0.58745\\p &= \boxed{58.745} \qquad \textbf{(B)}\end{aligned}$$

**15.21.** For a uniform, the mean is the midpoint of the interval, or $(1 + a)/2$. The variance is the interval length squared over 12, or $(a - 1)^2/12$. Using the given relationship between the mean and the variance,

$$\begin{aligned}\frac{1+a}{2} &= \frac{(a-1)^2}{2}\\(a-1)^2 - a - 1 &= 0\\a^2 - 3a &= 0\\a &= 0, \boxed{3}\end{aligned}$$

0 is excluded since $a > 1$. **(B)**

**15.22.** The variance of the hospitalization charge is $c^2/12 = (4\sqrt{3})^2 = 48$. So $c = 24$

The variance of the surgery charge is

$$\frac{(3c-18)^2}{12} = \frac{54^2}{12}$$

Therefore, the standard deviation of the surgery charge is $54/\sqrt{12} = \boxed{15.588}$. **(D)**

**15.23.** The variance of a uniform random variable on $[0, a]$ is $a^2/12$. Using this:

$$\begin{aligned}\frac{b}{\sqrt{12}} &= 9.60\\b &= 9.60\sqrt{12}\\\frac{2b-6}{\sqrt{12}} &= 19.2 - \frac{6}{\sqrt{12}} = \boxed{17.46795} \qquad \textbf{(C)}\end{aligned}$$

**15.24.** The probability that $X$ is between 8 and 11 is 0.4, so $X$'s range is $(11 - 8)/0.4 = 7.5$. The variance is the range squared divided by 12, which here is $7.5^2/12 = \boxed{4.6875}$. **(B)**

**15.25.** The variance of a single loss is calculated by conditioning on loss greater than 240 and using the conditional variance formula: Let $L$ be the loss and $P$ the excess over 240. Let $I$ be the indicator of loss being over 240.

$$\operatorname{Var}(P) = \mathbf{E}[\operatorname{Var}(P \mid I)] + \operatorname{Var}(\mathbf{E}[P \mid I]) = \mathbf{E}[0, 240^2/12] + \operatorname{Var}(0, 120) = 240^2/24 + 0.5^2(120^2) = 6000$$

The variance of the payment will be the percentage squared times 6000. So the percentage should be $\sqrt{1/3} = \boxed{0.57735}$. **(C)**

**15.26.** Suppose the interval is of length $l$. Then the interquartile range is $0.5l$ and the variance is $l^2/12$, so the standard deviation is $l/(2\sqrt{3})$. Dividing the standard deviation by the interquartile range, the $l$s cancel and we get $2/(2\sqrt{3}) = 1/\sqrt{3}$. **(A)**

**15.27.** The mean is the median. If the median is 12 and the 75th percentile is 18, since $0.75 - 0.5 = 0.25$, the difference $18 - 12$ is one-fourth of the range; the range is 24. The variance is range squared over 12, $24^2/12 = \boxed{48}$. **(C)**

**15.28.** Let $X$ be the loss. For deductible $d$, the expected value of a claim payment, using the double expectation formula, is

$$\mathbf{E}[\max(0, X-d)] = \Pr(X \le d)(0) + \Pr(X > d)\left(\frac{1-d}{2}\right) = \frac{(1-d)^2}{2}$$

Setting this equal to 0.245, we get $1-d = 0.7$. Thus $d = 0.3$. Using the conditional variance formula, with $I$ the indicator on a loss being greater than 0.3,

$$\begin{aligned}\operatorname{Var}(X) &= \mathbf{E}[\operatorname{Var}(X \mid I)] + \operatorname{Var}(\mathbf{E}[X \mid I]) \\ &= \mathbf{E}[0, 0.7^2/12] + \operatorname{Var}(0, 0.35) \\ &= \frac{0.7^3}{12} + (0.3)(0.7)(0.35^2) = \boxed{0.054308} \qquad \textbf{(B)}\end{aligned}$$

**15.29.** From the given expectation, we deduce that the midpoint is 3. From the given standard deviation, we deduce that the range is $\sqrt{12}$. The minimum is the midpoint minus half the range:

$$3 - 0.5\sqrt{12} = \boxed{1.2679} \qquad \textbf{(B)}$$

**15.30.** Let $A_k$ be the event that the family has exactly $k$ members and $B$ the event that 5 members ride the roller coaster. Then by Bayes' Theorem,

$$\Pr(A_6 \mid B) = \frac{\Pr(B \mid A_6)\Pr(A_6)}{\Pr(B)}$$

$\Pr(B \mid A_6) = 1/6$ and $\Pr(A_6) = \frac{8-6}{28} = \frac{1}{14}$. To calculate $\Pr(B)$, we calculate

$$\begin{aligned}&\Pr(B \mid A_5)\Pr(A_5) + \Pr(B \mid A_6)\Pr(A_6) + \Pr(B \mid A_7)\Pr(A_7) \\ &= \frac{1}{5}\left(\frac{3}{28}\right) + \frac{1}{6}\left(\frac{2}{28}\right) + \frac{1}{7}\left(\frac{1}{28}\right) \\ &= 0.038435\end{aligned}$$

The answer is

$$\Pr(A_6 \mid B) = \frac{\frac{1}{6}\left(\frac{1}{14}\right)}{0.038435} = \boxed{0.309735} \qquad \textbf{(E)}$$

**15.31.** Let $(R, D)$ be the number of Republicans and number of Democrats. There are 6 possibilities: (0,0), (0,1), (0,2), (1,0), (1,1), (2,0). When $R = D$ nothing is added to the expectation, so we can ignore those two. Probabilities of the other four are

$$\begin{aligned}p(0,1) &= \frac{\binom{5}{1}\binom{2}{1}}{\binom{10}{2}} = \frac{10}{45} \\ p(0,2) &= \frac{\binom{2}{2}}{\binom{10}{2}} = \frac{1}{45} \\ p(1,0) &= \frac{\binom{5}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{15}{45} \\ p(2,0) &= \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{3}{45}\end{aligned}$$

The expected absolute difference is

$$\frac{10}{45} + \frac{15}{45} + 2\left(\frac{1}{45} + \frac{3}{45}\right) = \frac{33}{45} = \boxed{\frac{11}{15}} \qquad \textbf{(D)}$$

**15.32.** The amount Actuary A reports, $X$, is uniformly distributed on $[0,10]$. The density of $X$ is $1/10$. The fourth moment of $X$ is

$$\int_0^{10} \frac{x^4\,dx}{10} = \frac{10^5}{5(10)} = 2000$$

The amount Actuary B reports, $Y$, assumes values 1,2,...,9 with probability 1/10 apiece, and 0 and 10 with probabilities 1/20. The fourth moment is

$$\frac{1}{10}(1^4 + 2^4 + \cdots + 9^4) + \frac{10^4}{20} = 2033.3$$

The absolute difference is $\boxed{33.3}$. **(B)**

**15.33.** The probability mass function of $N$ is 1/5 at each of the five points. The expected value of $N^2 + N + 1$ is

$$\mathbf{E}[N^2 + N + 1] = \frac{\sum_{n=1}^{5}(n^2 + n + 1)}{5} \qquad (*)$$

If you know the formulas

$$\sum_{n=1}^{k} n = \frac{k(k+1)}{2}$$
$$\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$$

then you can quickly calculate

$$\frac{\sum_{n=1}^{5}(n^2 + n + 1)}{5} = \frac{(5)(6)(11)/6 + (5)(6)/2 + 5}{5} = \frac{55 + 15 + 5}{5} = \boxed{15} \qquad \textbf{(E)}$$

If you don't know those formulas, you'd add up the five numbers in the expression in (*): $(3+7+13+21+31)/5 = \boxed{15}$.

# *Practice Exams*

The six practice exams provided here have questions similar to the SOA 328 and the exercises of this manual. They provide you an opportunity to see how well you know the material. You will be able to identify weaknesses and fix them before you take the real thing. I believe their level of difficulty is about the same as the real exam, but the later exams are somewhat more difficult than the earlier ones.

# Practice Exam 1

**1.** Six actuarial students are all equally likely to pass Exam P. Their probabilities of passing are mutually independent. The probability all 6 pass is 0.24.

Calculate the variance of the number of students who pass.

(A) 0.98 (B) 1.00 (C) 1.02 (D) 1.04 (E) 1.06

**2.** The number of claims in a year, $N$, on an insurance policy has a Poisson distribution with mean 0.25. The numbers of claims in different years are mutually independent.

Calculate the probability of 3 or more claims over a period of 2 years.

(A) 0.001 (B) 0.010 (C) 0.012 (D) 0.014 (E) 0.016

**3.** Claim sizes on an insurance policy have the following distribution:

$$F(x) = \begin{cases} 0, & x \le 0 \\ 0.0002x & 0 < x < 1000 \\ 0.4, & x = 1000 \\ 1 - 0.6e^{-(x-1000)/2000} & x > 1000 \end{cases}$$

Calculate expected claim size.

(A) 1500 (B) 1700 (C) 1900 (D) 2100 (E) 2300

**4.** An actuary analyzes weekly sales of automobile insurance, $X$, and homeowners insurance, $Y$. The analysis reveals that $\operatorname{Var}(X) = 2500$, $\operatorname{Var}(Y) = 900$, and $\operatorname{Var}(X+Y) = 3100$.

Calculate the correlation of $X$ and $Y$.

(A) $-0.2$ (B) $-0.1$ (C) 0 (D) 0.1 (E) 0.2

**5.** Several teams of two students compete in a math competition. In the competition, each student is given a set of problems to complete in 1 hour.

Let $F(m, n)$ represent the joint cumulative distribution function for the number of question completed by each student in a team in an hour.

Determine which of the following represents the probability that both students complete at least 5 questions in an hour.

(A) $1 - F(5,5)$
(B) $1 - F(5,0) - F(0,5)$
(C) $1 - F(5,\infty) - F(\infty,5)$
(D) $1 - F(5,\infty) - F(\infty,5) + F(5,5)$
(E) $1 - F(5,\infty) - F(\infty,5) + F(\infty,\infty)$

**6.** The probability of rain each day is the same, and occurrences of rain are mutually independent.

The expected number of non-rainy days before the next rain is 4.

Calculate the probability that the second rain will not occur before 7 non-rainy days.

(A) 0.11 (B) 0.23 (C) 0.40 (D) 0.50 (E) 0.55

**7.** On a certain day, you have a staff meeting and an actuarial training class. Time in hours for the staff meeting is $X$ and time in hours for the actuarial training session is $Y$. $X$ and $Y$ have the joint density function

$$f(x,y)=\begin{cases}\dfrac{3x+y}{250} & 0\le x\le 5,\ 0\le y\le 5\\ 0, & \text{otherwise}\end{cases}$$

Calculate the expected total hours spent in the staff meeting and actuarial training class.

(A) 3.33 (B) 4.25 (C) 4.67 (D) 5.17 (E) 5.83

**8.** In a small metropolitan area, annual losses due to storm and fire are assumed to be independent, exponentially distributed random variables with respective means 1.0 and 2.0.

Calculate the expected value of the maximum of these losses.

(A) 2.33 (B) 2.44 (C) 2.56 (D) 2.67 (E) 2.78

**9.** A continuous random variable $X$ has the following distribution function:

$$F(x)=\begin{cases}0 & x\le 0\\ 0.2 & x=3\\ 0.4 & x=8\\ 0.7 & x=16\\ 1 & x\ge 34\end{cases}$$

For $0<x<34$ not specified, $F(x)$ is determined by linear interpolation between the nearest two specified values.

Calculate the $80^{\text{th}}$ percentile of $X$.

(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

**10.** The side of a cube is measured with a ruler. The error in the measurement is uniformly distributed on $[-0.2, 0.2]$.

The measurement is 4.

Calculate the expected volume of the cube.

(A) 63.84 (B) 63.92 (C) 64.00 (D) 64.08 (E) 64.16



*Exam questions continue on the next page . . .*

**11.** For each sale of insurance, let $X$ be the number of days to complete underwriting and let $Y$ be the number of days to complete the sale. The distribution of $X$ and $Y$ is given in the following table:

| | $Y$ | | | | | |
|---|---|---|---|---|---|---|
| $X$ | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 0.04 | 0.06 | 0.05 | 0 | 0 | 0 |
| 2 | 0 | 0.07 | 0.10 | 0.08 | 0 | 0 |
| 3 | 0 | 0 | 0.09 | 0.15 | 0.12 | 0 |
| 4 | 0 | 0 | 0 | 0.05 | 0.10 | 0.09 |

Calculate the probability that underwriting takes 2 days or less given that completing the sale takes 5 days or less.

(A) 0.54 (B) 0.57 (C) 0.60 (D) 0.63 (E) 0.66

**12.** An insurance coverage provides for an insurance payment of 80% of the loss. The policy limit is 1000. The limit is applied before the multiplication by 80%.

Losses follow an exponential distribution with mean 800.

Calculate the expected payment per loss.

(A) 457 (B) 464 (C) 471 (D) 478 (E) 485

**13.** The quality of drivers is measured by a random variable $\Theta$. This variable is uniformly distributed on $[0, 1]$.

Given a driver of quality $\Theta = \theta$, annual claims costs for that driver have the following distribution function:

$$F(x \mid \theta) = \begin{cases} 0 & x \le 100 \\ 1 - \left(\dfrac{100}{x}\right)^{2+\theta} & x > 100 \end{cases}$$

Calculate the expected annual claim costs for a randomly selected driver.

(A) 165 (B) 167 (C) 169 (D) 171 (E) 173

**14.** $X$ represents losses on an auto liability policy due to bodily injury.

$Y$ represents losses on an auto liability policy due to property damage.

$X$ and $Y$ are independent and are both normally distributed with the following means and variances:

$$\mathbf{E}[X] = 150 \qquad \mathbf{E}[Y] = 50$$
$$\mathrm{Var}(X) = 5{,}000 \qquad \mathrm{Var}(Y) = 400$$

Calculate the probability that total losses, $X + Y$, are less than 250.

(A) 0.571 (B) 0.626 (C) 0.680 (D) 0.716 (E) 0.752



*Exam questions continue on the next page . . .*

**15.** On an insurance coverage, the number of claims submitted has the following probability function:

| Number of claims | Probability |
|---|---|
| 0 | 0.30 |
| 1 | 0.25 |
| 2 | 0.25 |
| 3 | 0.20 |

The size of each claim has the following probability function:

| Size of claim | Probability |
|---|---|
| 10 | 0.5 |
| 20 | 0.3 |
| 30 | 0.2 |

Claim sizes are independent of each other and of claim counts.

Calculate the mode of the sum of claims.

(A) 0 (B) 10 (C) 20 (D) 30 (E) 40

**16.** Among commuters to work:

i) 62% use a train.
ii) 25% use a bus.
iii) 18% use a car.
iv) 16% use a train and a bus.
v) 10% use a train and a car.
vi) 8% use a bus and a car.
vii) 2% use a train, a bus, and a car.

Calculate the proportion of commuters who do not use a train, a bus, or a car.

(A) 0.25 (B) 0.27 (C) 0.29 (D) 0.31 (E) 0.33

**17.** Two fair dice are tossed.

Calculate the probability that the numbers of the dice are even, given that their sum is 6.

(A) 1/3 (B) 2/5 (C) 1/2 (D) 3/5 (E) 2/3

**18.** In a box of 100 machine parts, 6 are defective. 5 parts are selected at random.

Calculate the probability that exactly 4 selected parts are not defective.

(A) 0.05 (B) 0.19 (C) 0.24 (D) 0.28 (E) 0.31



*Exam questions continue on the next page . . .*

**19.** Let $X$ and $Y$ be discrete random variables with joint probability function

$$p(x,y) = \begin{cases} \dfrac{x+y}{18} & (x,y) = (1,1),(1,2),(2,1),(2,4),(3,1) \\ 0, & \text{otherwise} \end{cases}$$

Determine the marginal probability function of $Y$.

(A) $p(x) = \begin{cases} \dfrac{5}{18} & y = 1 \\ \dfrac{1}{2} & y = 2 \\ \dfrac{2}{9} & y = 3 \end{cases}$

(B) $p(x) = \begin{cases} \dfrac{5}{18} & y = 1 \\ \dfrac{1}{2} & y = 2 \\ \dfrac{2}{9} & y = 4 \end{cases}$

(C) $p(x) = \begin{cases} \dfrac{1}{2} & y = 1 \\ \dfrac{1}{6} & y = 2 \\ \dfrac{1}{3} & y = 4 \end{cases}$

(D) $p(x) = \begin{cases} \dfrac{3}{5} & y = 1 \\ \dfrac{1}{5} & y = 2 \\ \dfrac{1}{5} & y = 4 \end{cases}$

(E) $p(x) = \begin{cases} \dfrac{2}{3} & y = 1 \\ \dfrac{1}{9} & y = 2 \\ \dfrac{2}{9} & y = 4 \end{cases}$

**20.** A blood test for a disease detects the disease if it is present with probability 0.95. If the disease is not present, the test produces a false positive for the disease with probability 0.03.

2% of a population has this disease.

Calculate the probability that a randomly selected individual has the disease, given that the blood test is positive.

(A) 0.05 (B) 0.39 (C) 0.54 (D) 0.73 (E) 0.97

**21.** A customer service representative in an insurance company handles calls from clients and agents.

- Let $X$ be the number of calls from agents. Mean of $X$ is 10 and standard deviation of $X$ is 5.
- Let $Y$ be the number of calls from clients. Mean of $Y$ is 35 and standard deviation of $Y$ is 60.

Based on a sample of 100 days, the 67$^{\text{th}}$ percentile of the total number of calls from agents and clients is 47.693.

Using the normal approximation, determine the approximate correlation factor between $X$ and $Y$, $\rho_{XY}$.

(A) 0.1 (B) 0.2 (C) 0.3 (D) 0.4 (E) 0.5

**22.** An actuarial department has 8 pre-ASA students and 5 students who are ASAs. To support a project to convert reserves to a new software system, the head of the project selects 4 students randomly.

Calculate the probability that at least 2 ASAs were selected.

(A) 0.39 (B) 0.46 (C) 0.51 (D) 0.56 (E) 0.63



*Exam questions continue on the next page . . .*

**23.** The amount of time a battery lasts, $T$, is normally distributed. The $20^{\text{th}}$ percentile of $T$ is 160 and the $30^{\text{th}}$ percentile is 185.

Calculate the $60^{\text{th}}$ percentile of $T$.

(A) 246 (B) 253 (C) 260 (D) 267 (E) 274

**24.** The number of houses on a block, $X$, follows a Poisson distribution with mean 10. Given $X$, the number of trees on a block, $Y$, follows a Poisson distribution with mean $X$.

Calculate $\text{Cov}(X, Y)$.

(A) 5.0 (B) 10.0 (C) 14.1 (D) 17.3 (E) 20.0

**25.** The side of a square is measured. The true length of the side is 10. The length recorded by the measuring instrument is normally distributed with mean 10 and standard deviation 0.1.

Calculate the expected area of the square based on the measurement recorded by the measuring instrument.

(A) 99.90 (B) 99.99 (C) 100.00 (D) 100.01 (E) 100.10

**26.** Losses on an insurance policy are modeled with a random variable with density function

$$f(x) = \begin{cases} cx^a & 0 < x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

The probability that a loss is less than 2, given that it is less than 3, is 0.5227.

Calculate the probability that a loss is greater than 1, given that it is less than 2.

(A) 0.64 (B) 0.67 (C) 0.70 (D) 0.73 (E) 0.76

**27.** Let $X$ be a random variable with the following distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2 & 0 \le x < 1 \\ 0.3 & 1 \le x < 2 \\ 0.5 + 0.1x & 2 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

Calculate $P[1 \le X \le 2]$.

(A) 0.1 (B) 0.2 (C) 0.3 (D) 0.4 (E) 0.5



*Exam questions continue on the next page . . .*

**28.** $X$ and $Y$ are discrete random variables with joint probability function

$$P(x,y) = \begin{cases} \left(\frac{1}{3}\right)^x & x \text{ a positive integer, and } y = 0 \\ \frac{1}{2}\left(\frac{1}{3}\right)^x & x \text{ a positive integer, and } y = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

You are given the formula

$$\sum_{x=0}^{\infty} c^x = \frac{c}{1-c}$$

Calculate $\mathbf{E}[XY]$.

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 3

**29.** A pair of dice is tossed.

Calculate the variance of the sum of the dice.

(A) $\dfrac{17}{3}$ (B) $\dfrac{35}{6}$ (C) 6 (D) $\dfrac{37}{6}$ (E) $\dfrac{19}{3}$

**30.** The daily number of visitors to a national park follows a Poisson distribution with mean 900.

Calculate the 80th percentile of the number of visitors in a day.

(A) 915 (B) 920 (C) 925 (D) 930 (E) 935

*Solutions to the above questions begin on page 541.*

# *Appendices*

# Appendix A. Solutions to the Practice Exams

## Answer Key for Practice Exam 1

| | |
|---|---|
| 1 | B |
| 2 | D |
| 3 | D |
| 4 | B |
| 5 | D |
| 6 | D |
| 7 | E |
| 8 | A |
| 9 | B |
| 10 | E |
| 11 | B |
| 12 | A |
| 13 | C |
| 14 | E |
| 15 | A |
| 16 | B |
| 17 | B |
| 18 | C |
| 19 | C |
| 20 | B |
| 21 | B |
| 22 | C |
| 23 | A |
| 24 | B |
| 25 | D |
| 26 | B |
| 27 | E |
| 28 | B |
| 29 | B |
| 30 | C |

## Practice Exam 1

1. [**Lesson** 17]

$$p^6 = 0.24$$
$$\mathrm{Var}(N) = 6p(1-p) = 6\sqrt[6]{0.24}(1-\sqrt[6]{0.24}) = \boxed{\mathbf{1.0012}} \qquad \textbf{(B)}$$

2. [**Lesson** 19] Over two years, the Poisson parameter is 2(0.25) = 0.5.

$$\Pr(N>2) = 1-p(0)-p(1)-p(2) = 1-e^{-0.5}\left(1+0.5+\frac{0.5^2}{2}\right) = \boxed{\mathbf{0.01439}} \qquad \textbf{(D)}$$

3. **[Lesson 7]** We will integrate the survival function.

$$\int_0^{1000} \big(1 - F(x)\big)\,\mathrm{d}x = \int_0^{1000} (1 - 0.0002x)\,\mathrm{d}x$$
$$= 1000 - 0.0001(1000^2) = 900$$
$$\int_{1000}^{\infty} \big(1 - F(x)\big)\,\mathrm{d}x = \int_{1000}^{\infty} 0.6e^{-(x-1000)/2000}\,\mathrm{d}x$$
$$= 0.6(2000) = 1200$$
$$\mathbf{E}[X] = 900 + 1200 = \boxed{\mathbf{2100}}$$

You can also do this using the double expectation formula. Given that $X < 1000$, it is uniform on [0,1000] with mean 500. Given that it is greater than 1000, the excess over 1000 is exponential with mean 2000, so the total mean is 1000 + 2000=3000.

$$\Pr(X < 1000) = 0.2$$
$$\Pr(X = 1000) = 0.2$$
$$\Pr(X > 1000) = 0.6$$
$$\mathbf{E}[X] = \Pr(X < 1000)(500) + \Pr(X = 1000)(1000) + \Pr(X > 1000)(3000)$$
$$= 0.2(500) + 0.2(1000) + 0.6(3000) = \boxed{\mathbf{2100}} \qquad \textbf{(D)}$$

4. **[Lesson 13]** $3100 = 2500 + 900 + 2\operatorname{Cov}(X,Y)$, so $\operatorname{Cov}(X,Y) = -150$. Then

$$\operatorname{Corr}(X,Y) = \frac{-150}{\sqrt{(2500)(900)}} = \boxed{\mathbf{-0.1}} \qquad \textbf{(B)}$$

5. **[Lesson 11]** If $A$ is the probability that the first student completes at least 5 and $B$ the probability that the second student completes at least 5, then we want

$$P[A \cap B] = P[A] + P[B] - P[A \cup B]$$

Note that $P[A] = 1 - F(5,\infty)$ and $P[B] = 1 - F(\infty,5)$. Also, $P(A \cup B) = 1 - F(5,5)$. Plugging these expressions into the formula for $P[A \cap B]$ yields **(D)**.

6. **[Lesson 18]** Let $p$ be the probability of rain. The negative binomial random variable for the first rain, with $k = 1$, has mean 4, so $(1-p)/p = 4$ and $p = 0.2$. We want the probability of less than 2 rainy days in the next 8 days. That probability is

$$\binom{8}{0}0.8^8 + \binom{8}{1}(0.2)(0.8^7) = \boxed{\mathbf{0.5033}} \qquad \textbf{(D)}$$