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Exam LTAM Study Manual

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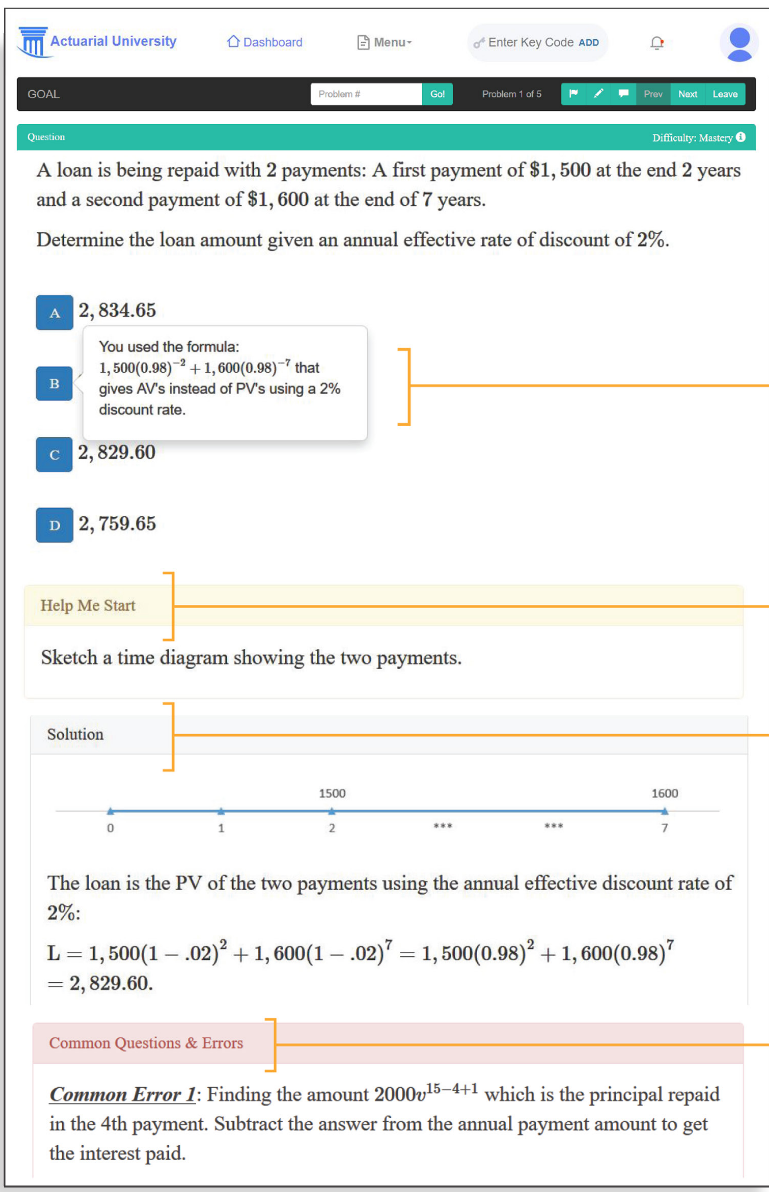

Flag for review, record notes & email the professor

Monitor difficulty level

Got it wrong? Often there is a simple reason why

Helpful strategies to get you started

Graphs and other solution techniques demonstrated when applicable

Commonly encountered errors

Start Your Exam Prep Today!

Preface

Welcome to Exam LTAM!

Exam LTAM is the exam in which life actuaries learn how to price and reserve for an insurance whose benefits may not be paid for a long time. This requires dealing with both probabilities of events and interest—the topics of exams P and FM.

Syllabus

The syllabus is posted at the following URL:

https://www.soa.org/globalassets/assets/files/edu/2021/fall/syllabi/ltam.pdf

The topics are

1. Survival models
2. Insurances
3. Annuities
4. Premiums
5. Reserves
6. Markov chains
7. Multiple decrement models
8. Multiple life models
9. Pensions
10. Profit tests
11. Estimating mortality rates and transition intensities

The textbook for the course is *Actuarial Mathematics for Life Contingent Risks* third edition. This is a college-style textbook. It is oriented towards practical application rather than exam preparation. Almost all exercises require use of spreadsheets or derivation of formulas.

In addition to the syllabus, you should read the introductory study note, at

https://www.soa.org/globalassets/assets/files/edu/2021/fall/intro-study-notes/2021-fall-intro-ltam.pdf

In paragraph 7, the note mentions that the Learning Outcomes, not the recommended text sections, comprise the syllabus and guide the exam committee when writing questions. Usually the text sections cover the learning outcomes, but in a few areas the text does not cover the learning outcomes directly. The readings corresponding to each lesson are listed at the beginning of the lesson, and occasionally it is mentioned that the lesson is not directly covered by the textbook.

The syllabus splits the material into six broad topics and states percentage ranges for them. These broad topics do not always match the organization of this manual or even the textbook.

Here is the distribution of questions by topic on MLC and LTAM exams, based on the organization of this manual, for the Spring 2012 through Fall 2013 exams, and the points per topic for the 2014 and later exams. Note that each question is classified based on the highest lesson required for it, so a question involving an asset share on universal

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life (there was one such question) would be classified as a universal life question. Thus a 0 does not indicate no questions on the topic on the exam. For example, the Fall 2016 exam had a question involving interest rate models (a topic not on the LTAM syllabus) applied to a multiple-life insurance, even though the table shows 0 for multiple life models.

		Questions				Points													
		MLC				MLC									LTAM				
Topic	Lessons	Spr 12	Fall 12	Spr 13	Fall 13	Spr 14	Fall 14	Spr 15	Fall 15	Spr 16	Fall 16	Spr 17	Fall 17	Spr 18	Fall 18	Spr 19	Fall 19	Spr 20	Fall 20
Introduction	2	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	0
Survival distributions	3–11	3	2	1	3	4	4	2	11	2	6	2	2	4	13	5	13	4	13
Insurances	13–18	0	2	2	1	3	2	6	4	13	6	13	6	0	6	6	4	2	3
Annuities	20–25	4	2	1	3	2	5	0	0	2	0	2	10	4	11	2	4	13	9
Premiums	27–35	5	2	3	6	17	18	20	10	22	17	17	15	10	9	13	16	7	13
Reserves	37–43	3	5	5	4	14	8	24	4	6	10	2	18	17	12	6	7	6	11
Markov chains	45–47	3	3	3	3	9	12	15	6	16	7	14	4	13	14	16	19	18	6
Multi-decrement models	49–54	2	1	2	0	2	2	6	8	0	9	4	11	0	2	2	6	12	4
Multiple life models	56–63	2	2	2	1	7	8	2	11	2	0	9	9	12	4	13	10	12	14
Estimating mortality rates	70–76	0	0	0	0	0	0	0	0	0	0	0	0	0	4	12	2	2	7
Pensions	65–67	1	0	1	1	11	11	8	6	15	14	13	11	13	14	12	4	12	12
Profit tests	68–69	5	4	3	2	20	26	13	26	18	16	20	10	6	6	8	9	8	4
Total		28	23	23	24	89	96	96	86	96	85	96	96	79	96	96	96	96	96

In this table, some of the topics dropped from the syllabus are omitted, so the total of questions and points does not always add up to the total on the exam (30 questions for Spring 2012, 25 questions for Fall 2012–Fall 2013, 96 points for all exams Spring 2014 and later).

As you can see in this table, weights on the topics have varied. The syllabus weight on pensions has increased twice within the last few years.

For the topics of this table, the distribution of questions on LTAM does not differ significantly from the distribution of questions on the former MLC, except for the addition of one new topic, estimating mortality rates. Weights can vary significantly since written answer questions have heavy weights and cover different topics in different years.

How LTAM differs from MLC

The following topics from MLC were dropped for LTAM:

1. Retrospective reserves
2. Policy alterations
3. Asset shares
4. Interest rate risk
5. Universal life
6. Participating products

The following topics were added to LTAM:

1. Mortality improvement models (Lessons 10 and 11)
2. Reserve recursions for multiple-state models (Section 47.4) and Woolhouse's formula for state-dependent annuities (Section 47.5).
3. Additional applications of Markov chain models (Lesson 48)

4. Retiree health benefits (Lesson 67)

5. Estimating mortality rates and transition intensities (Lessons 70–76)

Other downloads from the SOA site

Tables

Download the tables you will be given on the exam. They will often be needed for the exercises. They are a link at the end of the introductory study note. They are at the following URL:

`https://www.soa.org/Files/Edu/2018/ltam-standard-ultimate-life-table.pdf`

The tables include the Standard Ultimate Life Table, the Standard Sickness-Death Model, the Standard Service Table, some interest functions, and the standard normal distribution function. They even include a very short formula sheet with 5 sets of formulas!

The SOA has specified rules for using the normal distribution table they supply: *Do not interpolate in the table.*

Simply use the nearest value. If you are looking for $\Phi(0.0244)$, use $\Phi(0.02)$. If you are given the cumulative probability $\Phi(x) = 0.8860$ and need x, use 1.21, the nearest x available.

Another set of tables, the tables from the former MLC, will be useful if you wish to work on old exam questions that use the Illustrative Life Table. You can find them at

`https://www.soa.org/Files/Edu/edu-2013-mlc-tables.pdf`

However, I have converted all pre-2012 exam questions to use the Standard Ultimate Life Table, and the SOA converted questions from 2012 and later when they incorporated them in their sample questions. So it is unlikely you'll need the Illustrative Life Table.

Notation and terminology note

The note found at

`https://www.soa.org/Files/Edu/2018/ltam-notation-note.pdf`

discusses the terminology used on exam LTAM. Often different textbooks use different names for the same concept. In almost all cases, the exam uses the terminology of *Actuarial Mathematics for Life Contingent Risks*. This manual uses the terminology that will be used on the exam.

Sample questions

The SOA provides a set of sample questions based on MLC exams 2012 and later, with solutions. At the end of each lesson in this textbook, you will find a list of multiple-choice sample questions related to the material of the lesson if there are any. The questions and solutions themselves are not included in this manual. Nor is there a list of written-answer sample questions; those questions usually integrate material from more than one lesson.

Old exam questions in this manual

There are about 760 original exercises in the manual and about 970 old exam questions. The old exam questions come from old Part 4, Part 4A, Course 150, Course 151 exams, 2000-syllabus Exam 3, Exam C, Exam M, and Exam MLC. However, very few questions from the 2012 and later MLC exams are given in the exercises, so you may use those exams or the SOA sample questions as final practice. At this writing, the SOA sample questions do not include written answer questions from the 2017 and later exams, so those offer additional practice.

SOA Part 4 in 1986 had morning and afternoon sessions. I indicate afternoon session questions with "A". The morning session had the more basic topics (through reserves), while the afternoon session had advanced topics (multiple lives, multiple decrements, etc.) Both sessions were multiple choice questions.

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SOA Course 150 from 1987 through 1991 had multiple choice questions in the morning and written answer questions in the afternoon. Since LTAM will include written answer questions, I've included all applicable written answer questions in the exercises.

The CAS Part 4A exams awarded varying numbers of points to questions; some are 1 point and some are 2 points. The 1 point questions are probably too easy for a modern exam, but they'll give you a little practice. The pre-1987 exams probably were still based on Jordan (the old textbook), but the questions I provided, while ancient, still have value. Similarly, the cluster questions on SOA Course 150 in the 1990s generally were awarded 1 point per question.

The first edition of *Actuarial Mathematics*, which was used until around 1997, had commutation f unctions. Some old exam questions from the period before 1997 used commutation functions. In some cases, I've adapted such questions for use without commutation functions. You will see some "based on" questions where I made this adaptation. Even though these questions still have a commutation function feel, they are still legitimate questions.

Although the CAS questions are limited to certain topics, are different stylistically, and are easier, they are a good starting point.

Course 151 is the least relevant to this subject. I've only included a small number of questions from 151 in the first lesson, which is background.

Back in 1999, the CAS and SOA created a sample exam for the then-new 2000 syllabus. This exam had some questions from previous exams but also some new questions, some of them not multiple choice. This sample exam was never a real exam, and some of its questions were defective. This sample exam is no longer available on the web. I have included appropriate questions from it. *Whenever an exercise is labeled 1999 C3 Sample, it refers to the 1999 sample, not the current list of sample questions.*

Questions from CAS exams given in 2005 and later are not included in this manual. There is a lot of better practice material available, so in order to make this manual a little less bulky, I do not provide solutions to old CAS 3, 3L, and LC exams from the manual.

Questions from old exams are marked xxx:yy, where xxx is the time the exam was given, with S for spring and F for fall followed by a 2-digit year, and yy is the question number. Sometimes xxx is preceded with SOA or CAS to indicate the sponsoring organization. From about 1986 to 2000, SOA exams had 3-digit numbers (like 150) and CAS exams were a number and a letter (like 4A). From 2000 to Spring 2003, the exams were jointly sponsored. There was a period in the 1990s when the SOA, while it allowed use of its old exam questions, did not want people to reveal which exam they came from. As a result, I sometimes had study notes for old exams in this period and could not identify the exam they came from. In such a case, I mark the question aaa-bb-cc:yy, where aaa-bb-cc is the study note number and yy is the question number. Generally aaa is the exam number (like 150), and cc is the 2-digit year the study note was published.

New for the third edition

For Fall 2021, the syllabus uses *Actuarial Mathematics for Life Contingent Risks* for the mortality estimation topic; previously it used *Loss Models*. The new textbook covers less than the old textbook did. Thus I had to replace many practice exam questions.

As of this writing, the SOA has not yet removed sample questions on the deleted topics.

I have also replaced and shortened the exercise list for Lesson 71. You may skip this lesson and its exercise list, since this is material that you probably won't be directly tested on, but if you choose to work out the exercises, you will learn some of the derivations of mortality rate estimates in the textbook— and some other ones!

The SOA added Woolhouse's formula for state-dependent annuities to the syllabus for Spring 2021. Some of the mortality estimation practice exam questions I removed were replaced with Woolhouse questions.

Characteristics of this exam

The exam will have 20 multiple choice questions worth 2 points apiece and 6–7 written answer questions totalling 56 points, for a total of 96 points. You will be given 4 hours to complete the exam, or 2.5 minutes per point. The multiple choice questions will be easy, and are meant to screen students. Only students who perform above a threshold score on the multiple choice part of the exam will have their written answer questions graded.

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Table 1: 14 Week Study Schedule for Exam LTAM

Subject	Lessons	Study Period	Hard/Long Lessons	Easy/Short Lessons
Survival Distributions	3–11	2 weeks	4,8,11	3
Insurances	13–18	1 week	13,17	18
Annuities	20–25	1 week	20,23	25
Premiums	27–35	1.5 weeks	29,34	
Reserves, Part I	37–39	1 week		
Reserves, Part II	40–43	1 week	41, 43	
Markov Chains	45–48	1 week	48	
Multiple Decrements	49–54	1 week	52	51
Multiple Lives	56–63	1.5 weeks		56,59
Pension	65–67	1.5 weeks		
Profit Tests	68–69	0.5 weeks		69
Estimation	70–76	1 week		70,71,72,73

For multiple choice questions, there is no penalty for guessing. Fill in all questions regardless of whether you have time to work out the question or not—you lose nothing and you may be lucky!

The answer choices on SOA exams are almost always specific answers, not ranges.

For written answer questions, you will be graded on your work as well as your final a nswer. This manual has many shortcuts, but to the extent they are not in the textbook, you may have to derive the shortcut before using it. Written answer questions will be in many parts. Although most of the points will involve calculation, some parts of the question may ask you to explain concept or to determine the effect of varying a parameter. Sometimes a written answer question may ask you to derive a formula. This manual provides derivations for most formulas; make sure you understand these derivations.

The future of LTAM, and of this manual

In the introductory note, the SOA says that its current intention is that the Fall 2021 exam will have the same paper-and-pencil format as previous exams. But “Please check back on the Updates page (https://www.soa.org/education/exam-req/syllabus-study-materials/edu-updates-exam-ltam/) for the final decision.”

This manual was written based on this intention. Even though all the other written answer exams have been moved to CBT and include spreadsheets, LTAM does not have spreadsheets. If this changes, the SOA may be more specific as to what they e xpect. If they decide to provide something Excel-like, then this manual may need a supplement discussing approximate integration and providing spreadsheet exercises.

Study schedule

Although this manual seems huge, much of it is exercises and practice exams. You do not have to do every exercise; do enough to gain confidence with the m aterial. With intense studying, you should be able to cover all the material in 4 months.

It is up to you to set up a study schedule. Different students will have different speeds and different constraints, so it’s hard to create a study schedule useful for everybody. However, I offer a sample 14-week study schedule, Table 1, as a guide. This study schedule omits the first lesson, which is p reliminary. The amount of time you spend on this lesson depends on the strength of your probability background. You may decide to skip it and refer to it as needed.

The study schedule lists lessons that are either long or hard, as well as those that are short or easy or just background, so that you may better allocate your study time within the study periods provided for each subject.

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Acknowledgements

I would like to thank the SOA and CAS for allowing me to use their old exam questions. I'd also like to thank Harold Cherry for suggesting this manual and for providing three of the pre-2000 SOA exams and all of the pre-2000 CAS exams I used.

The creators of TEX, LATEX, and its multitude of packages all deserve thanks for making possible the professional typesetting of this mathematical material.

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Errata

Please report all errors you find in these notes to the author. You may send them to the publisher at mail@studymanuals.com or directly to me at errata@aceyourexams.net.

An errata list will be posted at errata.aceyourexams.net. Check this errata list frequently.

Flashcards

Many students find flashcards a useful tool for learning key formulas and concepts. ASM flashcards, available from the same distributors that sell this manual, contain the formulas and concepts from this manual in a convenient deck of cards. The cards have cross references, usually by page, to the manual.

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Lesson 13

Insurance: Annual and $1/m$thly—Moments

Reading: *Actuarial Mathematics for Life Contingent Risks* 3rd edition 4.1–4.3, 4.4.2, 4.4.3, 4.4.5–4.4.8

13.1 Review of Financial Mathematics

Since we are going to deal with present values, let's review the notation and formulas from Financial Mathematics that we will need. Let i be the interest rate. Then

- $d = i/(1+i)$ is the rate of discount.
- $v = 1/(1+i) = 1 - d$ is the discounting rate. A formula relating i, d, and v is $d = iv$.
- $\delta = \ln(1+i)$ is the continuously compounded interest rate corresponding to i. We have $e^{\delta} = 1+i$, $e^{-\delta} = v$, and $v^t = e^{-\delta t}$.
- $a_{\overline{n}|}$ is the present value of an n-year immediate certain annuity,[1] one that pays 1 per year at the end of each year for n years. The formula for it is
$$a_{\overline{n}|} = \frac{1-v^n}{i}$$
- $\ddot{a}_{\overline{n}|}$ is the present value of an n-year certain annuity-due, one that pays 1 per year at the beginning of each year for n years. The formula for it is
$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$
- $\bar{a}_{\overline{n}|}$ is the present value of an n-year continuous certain annuity, one that pays at a rate of 1 per year continuously. The formula for it is
$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$$
- $s_{\overline{n}|}$ is the cumulative value of an annuity, the value at the end of n years. The symbol may be decorated with double-dot or bar to represent "due" or "continuous", but the value is taken at the end of n years regardless. Thus it is $(1+i)^n$ times the corresponding "a" value; $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$, for example.
- The superscript (m) indicates payments on an $1/m$thly basis. For example, $\ddot{a}^{(4)}_{\overline{n}|}$ is an annuity-due paying once every three months at a rate of 1 per year, so it pays 0.25 every three months, starting immediately.

13.2 Moments of annual insurances

A life insurance pays a benefit to the insured policyholder upon death. The benefit may be payable whenever the insured dies, or may be payable only if the insured dies within a fixed number of years, or may be payable only if the insured dies after a fixed number of years. The benefit may vary depending on when death occurs.

We are interested in calculating the expected value of the present value of the benefit. This is the amount we would put aside today to fund the benefit. For a large group of independent policyholders, if we set aside the

[1] In this course, "annuity" means life annuity when not specified otherwise. Life annuities will be discussed later in the course. The annuity you learned about in your Financial Mathematics course will be called a certain annuity or an annuity-certain to distinguish it from the annuities of this course.

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expected present value of the benefit for each policyholder, the law of large numbers tells us that we will have approximately enough money to pay all the benefits when they become due. We would also like to calculate the variance of the present value of the benefit. Then we can use the normal approximation to determine the size of the fund that has a specific probability of being adequate to pay all the benefits.

The expected present value of the benefit is such an important concept that there are a lot of names for it. It is sometimes called the *actuarial present value* of the insurance. Another name for the expected present value is the *net single premium*, since this is the lump sum that a life insurance company would charge for providing such an insurance if no provision for expenses and profit was made. In the pre-2012 textbook for this exam, it is called the "single benefit premium", and you will find this terminology on old exams. In this manual, we will usually use the term "expected present value". On exams, any of these terms may be used, so we'll use "actuarial present value" on rare occasions to remind you that it has the same meaning as "expected present value".

Real life insurance pays the benefit soon after the death of the policyholder. However, in this lesson, we will discuss insurances that pay at the end of the year of death, rather than at the moment of death. This makes the random variable for the present value of the benefit discrete, and allows us to use life tables to calculate the expected value. We will sometimes call this insurance "discrete". In the exercises, some questions refer to "fully discrete" insurances. For the meantime, ignore the word "fully"; we will define it when we discuss premiums. In Lesson 18, we will discuss how to adjust the expected value of a discrete insurance to take into account payment at the moment of death.

The expected value of a discrete random variable is the sum of the probabilities of each possible outcome times the value of the random variable in each outcome. Here's a simple example of the calculation of the expected present value of an insurance:

> An insurance contract on a person age 40 will pay \$1000 at the end of a year to that person's estate if that person dies during the year.

Let the random variable Z be the present value of the payment made to the estate. It is equal to $1000v$ if death occurs within a year, 0 otherwise. The expected present value of the insurance is $1000v$ times the probability of death within a year; in other words, $\mathbf{E}[Z] = 1000vq_{40}$.

This logic can be generalized. For an insurance on (x), let b_k be the benefit paid at the end of year k for death during year k. Let Z be the present value random variable for the insurance. Then

$$\mathbf{E}[Z] = \sum_{k=0}^{\infty} b_{k+1}\, v^{k+1}\, {}_{k|}q_x = \sum_{k=0}^{\infty} b_{k+1}\, v^{k+1}\, {}_kp_x\, q_{x+k} \tag{13.1}$$

$$\mathbf{E}[Z^2] = \sum_{k=0}^{\infty} b_{k+1}^2\, v^{2(k+1)}\, {}_{k|}q_x = \sum_{k=0}^{\infty} b_{k+1}^2\, v^{2(k+1)}\, {}_kp_x\, q_{x+k} \tag{13.2}$$

EXAMPLE 13A An insurance on [70] pays 1000 at the end of the year of death if [70] dies after 1 year but not after 3 years.

You are given:

(i) Mortality is based on the following 2-year select-and-ultimate table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
70	0.05	0.07	0.10	72
71	0.06	0.08	0.12	73

(ii) $i = 0.04$

(iii) Z is the present value random variable for the insurance.

Calculate $\mathbf{E}[Z]$.

SOLUTION: The answer, by formula (13.1), is

$$\mathbf{E}[Z] = 1000\left(\frac{{}_{1|}q_{[70]}}{1.04^2} + \frac{{}_{2|}q_{[70]}}{1.04^3}\right)$$

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We have

$$ {}_{1|}q_{[70]} = p_{[70]}\, q_{[70]+1} = (0.95)(0.07) = 0.0665 $$
$$ {}_{2|}q_{[70]} = p_{[70]}\, p_{[70]+1}\, q_{72} = (0.95)(0.93)(0.10) = 0.08835 $$

So the answer is

$$ \mathbf{E}[Z] = 1000\left(\frac{0.0665}{1.04^2} + \frac{0.08835}{1.04^3}\right) = 1000(0.061483 + 0.078543) = \boxed{\mathbf{140.03}}. $$

□

To calculate the variance of the present value of an insurance, we use the formula

$$ \operatorname{Var}(Z) = \mathbf{E}[Z^2] - \mathbf{E}[Z]^2 $$

Example 13B Assume the same mortality and interest as the previous example. An insurance on [70] pays a benefit of 1000 at the end of the year if death occurs in the first year and 2000 at the end of the year if death occurs in the second year. Let Z be the payment random variable for this insurance.

Calculate $\operatorname{Var}(Z)$.

Solution: The first moment is

$$ \mathbf{E}[Z] = 1000\left(\frac{0.05}{1.04}\right) + 2000\left(\frac{0.0665}{1.04^2}\right) = 48.07692 + 122.96598 = 171.04290 $$

When calculating the second moment, the benefit as well as the interest factor must be squared.

$$ \mathbf{E}[Z^2] = 1000^2\left(\frac{0.05}{1.04^2}\right) + 2000^2\left(\frac{0.0665}{1.04^4}\right) = 46{,}227.81 + 227{,}377.91 = 273{,}605.72 $$
$$ \operatorname{Var}(Z) = 273{,}605.72 - 171.04290^2 = \boxed{\mathbf{244{,}350.05}} $$

□

13.3 Standard insurances and notation

We will discuss several standard types of insurance. For each one, if the benefit is 1, there is a standard symbol in International Actuarial Notation (IAN) for the expected present value. Here is a list of standard types of insurance coverage having standard notation to describe their expected present value:

Whole life insurance A whole life insurance of 1 on (x) pays 1 whenever death occurs. The standard symbol for the expected present value of whole life insurance payable at the end of the year of death is A_x. The random variable Z for a whole life insurance is v^{K_x+1}.

Term life insurance A term life insurance of 1 on (x) for n years pays 1 if death occurs within n years, 0 otherwise. The standard symbol for the expected present value of term life insurance payable at the end of the year of death is $A^{1}_{x:\overline{n}|}$. The 1 over the x indicates that the payment is made only if (x) expires before the n-year certain period. The random variable Z for a term life insurance is v^{K_x+1} if $K_x < n$, 0 otherwise.

Deferred whole life insurance An n-year deferred life insurance on (x) pays 0 if death occurs within n years, 1 otherwise. The standard symbol for the expected present value of deferred insurance payable at the end of the year of death is ${}_{n|}A_x$. The random variable Z for a deferred life insurance is v^{K_x+1} if $K_x \ge n$, 0 otherwise.

The sum of a term life insurance for n years and an n-year deferred life insurance is a whole life insurance.

Deferred term insurance It is possible to combine deferral and term. The expected present value of an insurance payable at the end of the year of death only if death occurs between times n and $n+m$ would be denoted by ${}_{n|}A^{1}_{x:\overline{m}|}$. The random variable Z is v^{K_x+1} if $n \le K_x < n+m$, 0 otherwise.

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Pure endowment An n-year pure endowment of 1 on (x) pays 1 at the end of n years if (x) survives to that point, 0 otherwise. There are two standard symbols for its expected present value. The insurance style symbol is $A_{x:\overset{\;1}{\overline{n}|}}$. The 1 over the n indicates that the payment is made only if the n-year certain period expires before (x). The other, annuity style, symbol is ${}_nE_x$. Get used to both symbols! The random variable Z for the expected present value of a pure endowment is v^n if $K_x \ge n$, 0 otherwise. The expected present value of an n-year pure endowment of 1 on (x) is ${}_np_x\, v^n$.

Endowment insurance An n-year endowment insurance of 1 on (x) pays 1 at the earlier of the death of (x) or n years. It is the sum of a term insurance for n years and an n-year pure endowment. The standard symbol for the expected present value of it is $A_{x:\overline{n}|}$. The random variable Z for endowment insurance is v^{K_x+1} if $K_x < n$, v^n otherwise.

The above descriptions are summarized in Table 13.1.

Table 13.1: Actuarial notation for standard types of insurance

Name	Present value random variable		Symbol for expected present value
Whole life insurance	v^{K_x+1}		A_x
Term life insurance	v^{K_x+1} 0	$K_x < n$ $K_x \ge n$	$A^{\;1}_{x:\overline{n}|}$
Deferred life insurance	0 v^{K_x+1}	$K_x < n$ $K_x \ge n$	${}_{n|}A_x$
Deferred term insurance[a]	0 v^{K_x+1} 0	$K_x < n$ $n \le K_x < n+m$ $K_x \ge n+m$	${}_{n|}A^{\;1}_{x:\overline{m}|}$
Pure endowment	0 v^n	$K_x < n$ $K_x \ge n$	$A_{x:\overset{\;1}{\overline{n}|}}$ or ${}_nE_x$
Endowment insurance	v^{K_x+1} v^n	$K_x < n$ $K_x \ge n$	$A_{x:\overline{n}|}$

[a] Another symbol that can be used is ${}_{n|m}A_x$.

Quiz 13-1 Which symbol would be used for the item we calculated in Example 13A?

Professor Geoffrey Crofts introduced the following notation: the present value random variable for an insurance by Z has the same decorations as the corresponding expected value. For example, $Z_{30:\overline{10}|}$ is the present value random variable for a 10-year endowment insurance on (30). While this notation can be useful, it is not standard actuarial notation, and is not used by *Actuarial Mathematics for Life Contingent Risks*, so it will not appear on the exam.

A useful formula for calculating higher moments of standard insurances is that the k^{th} moment of a standard insurance equals the first moment calculated at k times the force of interest. This works whenever the death benefit in all years is either 0 or 1. This method is called *the rule of moments*. The concept of the first moment calculated at k times the force of interest is so useful, we will introduce a symbol for it. Whenever the presuperscript of an actuarial symbol is k, that means that the item is calculated at k times the force of interest.[2] For example, ${}^2A_x$ is the expected present value of a whole life insurance of 1 on (x) calculated at twice the force of interest. Then if Z is the present value random variable for an n-year endowment insurance of 1,

$$\operatorname{Var}(Z) = {}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2 \tag{13.3}$$

[2] However, as indicated in Section 37, a presuperscript means something else for the actuarial symbol V.

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A presuperscript of 2 *does not* represent the second moment; it merely represents doubling δ, the force of interest. ${}^2A_x$ happens to be the second moment of the random variable for which A_x is the first moment, but we will see later that a corresponding statement is not true for annuity symbols.

Doubling the force of interest is not the same as doubling the rate of interest. It actually "squares" the rate of interest. For example, if δ is $\ln 1.06$ (corresponding to $i = 0.06$), then 2δ is $2\ln 1.06 = \ln 1.06^2 = \ln 1.1236$, which corresponds to $i = 0.1236$. In general, if δ, i, d, and v correspond to each other, and we use a presuperscript of 2 on i, d, and v to indicate doubling the force of interest, then

$$\begin{aligned} {}^2i &= 2i + i^2 \\ {}^2d &= 2d - d^2 \\ {}^2v &= v^2 \end{aligned} \tag{13.4}$$

13.4 Standard Ultimate Life Table

On the exam, you may be asked to calculate insurance moments using the Standard Ultimate Life Table and $i = 0.05$.

The Standard Ultimate Life Table has the basic functions l_x and q_x, which can be used to calculate moments of a term insurance of short duration regardless of the interest rate assumption. In addition, it has single life actuarial functions A_x, ${}^2A_x$, and ${}_kE_x$ for $k = 5, 10, 20$, all at 5% interest, which lets you calculate long-term insurances without having to add up a lot of numbers. It also has $A_{x:\overline{10}|}$ and $A_{x:\overline{20}|}$, which allow you to compute 10- and 20- year endowment insurances. The A_x column is used for first moments, and the ${}^2A_x$ column is used for second moments. The ${}^2A_x$ column can also be used for first moments at $1.05^2 - 1 = 0.1025$ interest, but it's rare you'd use it for that purpose.

To calculate expected present values for term life insurances of 10 or 20 years, calculate them as the difference between an endowment insurance and a pure endowment for the same period. For other term insurances, deferred life insurances, and endowment insurances other than 10 and 20 years, use the pure endowment columns in conjunction with the insurance columns. You must relate these types of insurance to whole life insurance. An n-year deferred life insurance on (x) is an n-year pure endowment providing a whole life insurance on $(x+n)$. An n-year term life insurance on (x) is a whole life insurance on (x) minus an n-year deferred life insurance on (x). An n-year endowment insurance on (x) is an n-year term insurance on (x) plus an n-year pure endowment on (x). In symbols:

$$A^{1}_{x:\overline{n}|} = A_{x:\overline{n}|} - {}_nE_x \tag{13.5}$$

$$ {}_{n|}A_x = {}_nE_x\, A_{x+n} \tag{13.6}$$

$$A^{1}_{x:\overline{n}|} = A_x - {}_{n|}A_x = A_x - {}_nE_x\, A_{x+n} \tag{13.7}$$

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x = A_x - {}_nE_x\, A_{x+n} + {}_nE_x \tag{13.8}$$

Exam questions will almost always set n equal to a multiple of 5. For $n = 5, 10, 20$, you are given ${}_nE_x$. For $n = 15$, use ${}_5E_x\,{}_{10}E_{x+5}$ or ${}_{10}E_x\,{}_5E_{x+10}$. You can similarly obtain ${}_{25}E_x$, ${}_{30}E_x$, and ${}_{40}E_x$ by multiplying two of the pure endowments in the table together. In the rare case of an exam question requiring an n not a multiple of 5, you will have to calculate ${}_np_x$ as a quotient l_{x+n}/l_x and then multiply by v^n.

These four formulas apply equally well to second moments as long as you square the benefits and double the force of interest. The variance can then be calculated from the first and second moments. To calculate the expected present value of a pure endowment at twice the force of interest, multiply the first moment by v^n, where n is the period of the pure endowment. For example, if we are using the Standard Ultimate Life Table with $i = 0.05$, a 10-year pure endowment on (40) at twice the force of interest at $i = 0.05$ is ${}_{10}E_{40}/1.05^{10}$, where ${}_{10}E_{40}$ from the Standard Ultimate Life Table is 0.60920. The result is 0.37400.

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Example 13C For a 15-year endowment insurance on a life currently age 60 with face amount of 1000 and benefit payable at the end of the year of death:

(i) Mortality follows the Standard Ultimate Life Table.
(ii) $i = 0.05$

Calculate the expected present value of the insurance.

Solution:

$$1000A_{60:\overline{15}|} = 1000A_{60} - 1000\,{}_{15}E_{60}\,A_{75} + 1000\,{}_{15}E_{60}$$
$${}_{15}E_{60} = {}_{5}E_{60}\,{}_{10}E_{65} = (0.76687)(0.55305) = 0.42412$$
$$1000A_{60:\overline{15}|} = 290.28 - 0.42412(508.68) + 424.12 = \boxed{\mathbf{498.66}}$$

On exams, a *special* insurance is one whose benefits are not level.

Example 13D A special insurance on (65) pays 1000 at the end of the year of death if death occurs between ages 70 and 80, and 600 at the end of the year of death if death occurs after age 80. Mortality follows the Standard Ultimate Life Table and $i = 0.05$.

Calculate the single net premium for this insurance.

Solution: There are several ways to view this insurance. One way is to view it as a 5-year deferred whole life insurance for 1000 minus a 15-year deferred whole life insurance for 400, and we will do it this way. Alternatively, it is a sum of a 5-year deferred 10-year term insurance of 1000 plus a 15-year deferred whole life insurance of 600. That alternative expresses it as a sum of two mutually exclusive insurances and is better for calculating variance (which is not required for this example).

The expected present value of a 5-year deferred whole life insurance of 1000 is a whole life insurance of 1000 on 70 discounted 5 years with mortality and interest, or $1000\,{}_5E_{65}\,A_{70}$. The Standard Ultimate Life Table provides ${}_kE_x$ for $k = 5$, 10, and 20. The 5-year deferred insurance's expected present value is

$$1000\,{}_{5|}A_{65} = 1000\,{}_5E_{65}\,A_{70} = 1000(0.75455)(0.42818) = 323.08$$

The expected present value of a 15-year deferred whole life insurance of 400 is $400\,{}_{15}E_{65}\,A_{80}$. To use the Standard Ultimate Life Table's values of ${}_kE_x$, we express ${}_{15}E_{65} = {}_5E_{65}\,{}_{10}E_{70}$. (You can also express it as ${}_{10}E_{65}\,{}_5E_{75}$.) *Whenever k is a multiple of 5 (as it will probably be on the exam), decompose ${}_kE_x$ into components found in the Standard Ultimate Life Table!* It is faster to use the values for ${}_kE_x$ in the SULT than to compute ${}_kE_x$ using l_x's and discounting at 5%. The 15-year deferred insurance's expected present value is

$$400\,{}_{15|}A_{65} = 400\,{}_5E_{65}\,{}_{10}E_{70}\,A_{80} = 400(0.75455)(0.50994)(0.59293) = 91.26$$

The single net premium for the insurance is $323.08 - 91.26 = \boxed{\mathbf{231.82}}$.

Quiz 13-2 For a 5-year term life insurance on (45), you are given:

(i) A benefit of 1000 is paid at the end of the year of death.
(ii) Mortality follows the Standard Ultimate Life Table.
(iii) $i = 0.05$.

Calculate the second moment of the present value of the benefit payment.

13.5 Moments for insurance under constant force of mortality and uniform mortality

We will derive formulas for moments of insurance under constant force of mortality and uniform mortality. However, it is rare that these assumptions will be used on exams in conjunction with a discrete insurance, and if they came up on a written answer question you'd have to rederive the formulas, so you may skip this section if you wish.

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Constant force of mortality

If mortality has constant force, or more generally if q_{x+k} is constant for all integral $k \geq 0$, then the sum to evaluate the moment is a geometric series.

Example 13E For a whole life insurance on (x), you are given

(i) The valuation interest rate is i.
(ii) $q_{x+k} = q$ is constant for integral $k \geq 0$.

Calculate the expected value and variance of the present value of the insurance.

Solution: The survival probabilities are ${}_kp_x = (1-q)^k$.

Let Z be the present value random variable. Using equation (13.1),

$$\begin{aligned} A_x &= \sum_{k=0}^{\infty} (1-q)^k q v^{k+1} \\ &= \frac{qv}{1-(1-q)v} \\ &= \frac{qv}{d+qv} \\ &= \boxed{\frac{q}{q+i}} \end{aligned} \tag{13.9}$$

For the second moment, double the force of interest.

$$\mathbf{E}[Z^2] = \frac{q}{q+2i+i^2}$$

Then the variance is ${}^2A_x - A_x^2$.

$$\operatorname{Var}(Z) = \boxed{\frac{q}{q+2i+i^2} - \left(\frac{q}{q+i}\right)^2}$$

□

Uniform mortality

Now consider the assumption that mortality is uniformly distributed, or more generally that ${}_{k|}q_x = {}_kp_x\, q_{x+k}$ is a constant c for all integral $k \geq 0$. For an insurance with benefit 1 for death between times m and n, equation (13.1) becomes

$$\mathbf{E}[Z] = \sum_{k=m}^{n-1} v^{k+1} c = c v^m a_{\overline{n-m}|}$$

Note that $c = 1/(\omega - x)$. For a whole life insurance, $n = \omega - x$ and $m = 0$.

Example 13F Age at death for (35) is uniformly distributed, with $\omega = 100$. You are given:

(i) $i = 0.05$
(ii) Z is the present value random variable for a 20-year endowment insurance on this life.

Calculate $\operatorname{Var}(Z)$.

Solution: $\omega - x = 65$. Z is the sum of a 20-year term insurance and a 20-year pure endowment. Let Z_1 be the present value random variable for the term insurance and Z_2 the present value random variable for the pure endowment. The expected value of Z_1 is

$$\mathbf{E}[Z_1] = \frac{a_{\overline{20}|}}{65} = \frac{1}{65}\,\frac{1-(1/1.05)^{20}}{0.05} = 0.19173$$

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Table 13.2: Expected present value under constant force and uniform mortality for insurances payable at the end of the year of death

Type of insurance	Expected present value under constant force	Expected present value under uniform distribution
Whole life	$\dfrac{q}{q+i}$	$\dfrac{a_{\overline{\omega-x}\rvert}}{\omega-x}$
n-year term	$\dfrac{q}{q+i}\left(1-(vp)^n\right)$	$\dfrac{a_{\overline{n}\rvert}}{\omega-x}$
n-year deferred life	$\dfrac{q}{q+i}(vp)^n$	$\dfrac{v^n a_{\overline{\omega-(x+n)}\rvert}}{\omega-x}$
n-year pure endowment	$(vp)^n$	$\dfrac{v^n\left(\omega-(x+n)\right)}{\omega-x}$

and the second moment is calculated using twice the force of interest, or squaring v.

$$\mathbf{E}[Z_1^2]=\frac{1}{65}\frac{1-(1/1.05)^{40}}{1.05^2-1}=0.12877$$

The expected value of Z_2 is ${}_{20}p_{35}\,v^{20}$, and ${}_{20}p_{35}=(100-55)/(100-35)=45/65$.

$$\mathbf{E}[Z_2]=\frac{45}{65}\frac{1}{1.05^{20}}=0.26092$$

The second moment is calculated by squaring v.

$$\mathbf{E}[Z_2^2]=\frac{45}{65}\frac{1}{1.05^{40}}=0.09834$$

Therefore, the variance of Z is

$$\begin{aligned}\mathbf{E}[Z]&=0.19173+0.26092=0.45265\\ \mathbf{E}[Z^2]&=0.12877+0.09834=0.22711\\ \operatorname{Var}(Z)&=0.22711-0.45265^2=\boxed{\mathbf{0.02222}}\end{aligned}$$ □

Table 13.2 summarizes the results of this section. However, these formulas are unlikely to appear on an exam, and are not in the textbook so you'd be expected to derive them on a written answer question.

Quiz 13-3 A whole life insurance on (60) pays a benefit of 1000 at the end of the year of death. You are given:

(i) $\mu_{60+t}=\begin{cases}0.02 & t\le 10\\ 1/(60-t) & 10<t<60\end{cases}$

(ii) $i=0.06$

Calculate the net single premium for this insurance.

13.6 Normal approximation

For a large group of insureds, the distribution of the present value of benefit payments can be approximated as a normal distribution with mean and variance as calculated by the methods of this lesson. When working out such

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problems, remember that the variance of a coverage of x on one insured is x^2 times the variance of a coverage of 1, but the variance of a coverage of 1 on n independent insureds is only n times the variance of a coverage of 1 on one insured.

Example 13G A group of 400 independent individuals age (40) each purchases a whole life insurance of 1000. You are given:

(i) Mortality follows the Standard Ultimate Life Table.
(ii) $i = 0.05$.

Using the normal approximation, calculate the size of the fund needed so that there is a 95% probability that the fund will be adequate to pay all benefits.

Solution: Let Z be the present value random variable for one insurance of 1.

$$\mathbf{E}[Z] = A_{40} = 0.12106$$
$$\mathbf{E}[Z^2] = {}^2A_{40} = 0.02347$$
$$\operatorname{Var}(Z) = 0.02347 - 0.12106^2 = 0.008814$$

Let Z^p be the present value random variable for the entire portfolio of 400 insurances of 1000. For 400 individuals and 1000 of insurance, the expected value is multiplied by (400)(1000) and the variance is multiplied by $(400)(10^6)$. Then

$$\mathbf{E}[Z^p] = 0.12106(400{,}000) = 48{,}424$$
$$\sqrt{\operatorname{Var}(Z^p)} = \sqrt{0.008814(400)(10^6)} = 1{,}878$$

The 95$^{\text{th}}$ percentile of the standard normal distribution is 1.645. By the normal approximation, the fund needed is $48{,}424 + 1.645(1{,}878) = \boxed{\mathbf{51{,}513}}$. □

13.7 $1/m$thly insurance

In addition to the concept of an insurance payable at the end of the year of death, there is the concept of an insurance payable at the end of a fraction of the year of death. An example would be an insurance payable at the end of the month of death. In *Actuarial Mathematics for Life Contingent Risks*, they use the precise name *$1/m$thly insurance* for this concept, where, for example, $m = 12$ if the insurance is payable at the end of the month of death, but most other textbooks and actuaries would call it an "$m$thly insurance". This should not confuse you, because exam questions will say "quarterly", "monthly", or "4 times a year", or something similar. The concept of $1/m$thly insurance is neither realistic (since commercially sold insurances pay soon after the moment of death) nor helpful for calculations (since mortality tables are by year, not by month). However, the concept will be useful when dealing with $1/m$thly annuities, which *are* a realistic product.

Let $K_x^{(m)}$ be the future lifetime of (x) rounded down to the next lowest $1/m$. Thus if (40) dies after 30.4 years, then $K_{40}^{(2)} = 30$, $K_{40}^{(4)} = 30.25$, and $K_{40}^{(12)} = 30\frac{1}{3}$. $A^{(m)}$ (with suitable subscripts) is the expected present value of an insurance that pays at the end of a period of $1/m$ years in which the death occurs. For example, $A_x^{(12)}$ is a whole life insurance that pays at the end of the month of death. Then $A_x^{(m)} = \mathbf{E}[v^{K_x^{(m)}+1/m}]$. Calculating this requires summing up probabilities of death within each $1/m$ year times an appropriate discount factor:

$$A_x^{(m)} = \sum_{k=0}^{\infty} v^{(k+1)/m} \; {}_{\frac{k}{m}|\frac{1}{m}}q_x$$

Higher moments can be calculated by using a multiple of the force of interest.

Example 13H You are given that $q_{45} = 0.01$ and $q_{46} = 0.02$. Mortality is uniformly distributed between integral ages and $i = 0.05$.

Calculate $A^{(2)}_{\overset{1}{45}:\overline{2|}}$.

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Solution: The formula for the answer sums up the probabilities of the payments at the end of 0.5 years, 1 year, 1.5 years, and 2 years times the present values of those payments.

$$A^{(2)}_{\stackrel{1}{45}:\overline{2}|} = v^{0.5}\,{}_{0.5}q_{45} + v\,{}_{0.5|0.5}q_{45} + v^{1.5}\,{}_{1|0.5}q_{45} + v^{2}\,{}_{1.5|0.5}q_{45}$$

Let's compute the qs.

$$\begin{aligned}
{}_{0.5}q_{45} &= {}_{0.5|0.5}q_{45} = 0.5q_{45} = 0.5(0.01) = 0.005\\
{}_{1|0.5}q_{45} &= {}_{1.5|0.5}q_{45} = 0.5p_{45}q_{46} = 0.5(0.99)(0.02) = 0.0099\\
A^{(2)}_{\stackrel{1}{45}:\overline{2}|} &= 0.005\left(1.05^{-0.5} + 1.05^{-1}\right) + 0.0099\left(1.05^{-1.5} + 1.05^{-2}\right) = \boxed{\mathbf{0.027822}}
\end{aligned}$$

□

Note

The concepts discussed in these lessons can be applied to non-life insurance situations. They can be applied to any situation in which a payment is made contingent on some event, with the present value of the payment depending on when the event occurs. A warranty that replaces a product if it fails within 5 years would be an example.

Exercises

Expected value

13.1. You are given that $i = 0$.

Which of the following expressions equals $A_{x:\overline{30}|}$?

(A) ${}_{30}p_x$ (B) ${}_{30}q_x$ (C) ${}_{30|}q_x$ (D) p_{x+30} (E) 1

13.2. You are given that $i = 0$.

Which of the following expressions equals $A^{\;1}_{x:\overline{30}|}$?

(A) ${}_{30}p_x$ (B) ${}_{30}q_x$ (C) ${}_{30|}q_x$ (D) p_{x+30} (E) 1

13.3. You are given that $i = 0$.

Which of the following expressions equals ${}_{10|}A_x$?

(A) 0 (B) ${}_{10}p_x$ (C) ${}_{10}q_x$ (D) ${}_{10|}q_x$ (E) 1

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Exercises continue on the next page . . .

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