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# Exam P Study Manual



**3<sup>rd</sup> Edition, 2<sup>nd</sup> Printing**

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# Introduction

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Welcome to Exam P! This manual will prepare you for the exam. You are given 29 bite-size chunks, each with explanations, techniques, examples, and lots of exercises. If you learn the techniques in these 29 lessons, you will be well-prepared for the exam.

A prerequisite to this exam is calculus. Heavy calculus techniques are not tested, but you must know the basics. A short review of integration techniques follows this introduction. If these look totally unfamiliar to you, you must review your calculus before proceeding further!

## Exercises in this manual

There are several types of exercises in this manual:

1. Questions from the SOA 330. These are probably the most representative of what will appear on your exam. The lower numbered ones come from the exams released between 2000 and 2003. The higher numbered ones are probably retired questions from the CBT (computer based testing) data bank, the question bank from which the questions on your CBT exam are taken from.
2. Questions from the five released exams 2000–2003 not contained in the SOA 330. Those exams tested both probability and calculus, so only about half the questions are relevant. Almost all of the relevant questions are in the SOA 330, but a small number of them are not in that list.
3. Questions from the 1999 Sample Exam. This sample exam was published before the first exam under the 2000 syllabus. The 2000 syllabus was a drastic syllabus change and was accompanied by a new exam question style. This sample exam reflects the new style of question, but most of the questions never appeared on a real exam, so the questions may not be totally realistic exam questions.
4. Questions from pre-2000 released exams. The probability syllabus was hardly different before 2000—it is extremely stable—but the style of questions was much different. Questions before 2000 tended to be stated purely mathematically, whereas starting in 2000 almost all questions have a practical sounding context. Thus a pre-2000 exam question might be:

$A$  and  $B$  are two events. You are given that  $P[A] = 0.8$ ,  $P[B] = 0.6$ , and  $P[A \cup B] = 0.9$ .

What is  $P[A \cap B]$ ?

The same question appearing on a 2000 or later exam would read:

A survey of cable TV subscribers finds:

- 80% of subscribers watch the Nature channel.
- 60% of subscribers watch the History channel.
- 90% of subscribers watch at least one of the Nature or History channels.

Calculate the percentage of subscribers that watch both the Nature and the History channels.

Notice, among other things, that the final line of the question is no longer a question; it is always a directive. Some of the post-2000 released exam questions still asked questions. When these questions were incorporated into the SOA 330, they changed the question to a directive. Questions are no longer used.

I was thinking of rewording all the pre-2000 questions in the current style. But I am not good at creating contexts. I'd probably just copy contexts from existing questions, and it would get boring. Even though the style of these old questions is different, they are based on the same material, so working them out will help you learn the material you need to know for this exam.

5. Original questions. I try to write my questions in the current style, but I often copy contexts from SOA 330 questions.

The solutions to exercises are not necessarily the best methods for solving them. Sometimes, a technique in a later lesson may provide a shortcut. This is particularly true for exercises in the earlier lessons using standard distributions as illustrations. For example, an exercise in an early lesson may ask for the mean of a conditional distribution involving an exponential; after learning in the exponential lesson that an exponential is memoryless, this sort of question may be trivial. *However, if you find a better way to do an exercise that does not involve techniques from later lessons, please contact the author at the same address as the one for sending errata.*

This manual has six practice exams. All the questions on these practice exams are original.

## Useful features of this manual

As noted above, there is a very brief calculus note right after this introduction, for reviewing integration techniques.

There is an index at the end of the manual. If you ever remember a term and don't remember where you saw it, refer to the index. If it isn't in the index and you think it is in the manual, contact the author so that he can add it to the index.

Before the index, there are four cross reference tables. First, a cross reference for the practice exams. Some students prefer to do the practice exam questions as additional practice after each lesson, rather than saving them for final review, and this cross reference table lets you do that.

Second and third, tables showing you where each of the SOA 330 questions is listed as an exercise in the manual. If you are interested in seeing my solution to each of these questions, these tables are the place to go.

Fourth, a table showing you where each of the released exam questions from 2000–2003 is listed as an exercise in the manual.

## SOA downloads

On the SOA website, you will find the syllabus for the exam. This syllabus has links to other useful material. You'll find links to the 5 released exams and the SOA 330 questions and answers.

There is an important link to *Risk and Insurance* study note, authored by Judy Feldman Anderson and Robert L. Brown. At this writing, the URL of the study note is <http://www.soa.org/files/pdf/P-21-05.pdf>. This note gives you background information on how insurance works. You will not be tested directly on this study note, but insurance is used very often as the context of a question, so you should understand basic insurance terminology.

This study note has some probability examples, so you won't understand it in its entirety if you have never studied probability. Still, you should read the nonmathematical parts (in bed, or at some other leisure time) as soon as you can, ignoring the mathematical examples, since the exercises in this manual often have insurance contexts. At the point indicated in the manual in Lesson 8, you will have all the probability background you need to understand the study note's mathematics, and should read the study note in its entirety.

Another important link is to the table of the cumulative distribution function of the standard normal distribution. At this writing, it is at <http://www.soa.org/files/pdf/P-05-05tables.pdf>. You do not need this table until Lesson 23, but you should download it by then. For your convenience, a standard normal distribution table is provided in Appendix B.

## New for this edition

SOA sample questions released in 2016–2018 have been added to this edition of the manual.

## Errata

Please report any errors you find. Reports may be sent to the publisher ([mail@studymanuals.com](mailto:mail@studymanuals.com)) or directly to me ([errata@aceyourexams.net](mailto:errata@aceyourexams.net)). *When reporting errata, please indicate which manual and which edition and printing you are*

*referring to!* This manual is the 3<sup>rd</sup> edition of the Exam P manual.

An errata list will be posted at <http://errata.aceyourexams.net>

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I thank Geoff Tims for proofreading the manual. As a result of his work, many mathematical errors were corrected and many unclear passages were clarified. I also thank Kristen McLaughlin for her comments and suggestions for improvements.

I thank Donald Knuth, the creator of  $\text{T}_{\text{E}}\text{X}$ , Leslie Lamport, the creator of  $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$ , and the many package writers and maintainers, for providing a typesetting system which allows such beautiful typesetting of mathematics and figures.

I thank the many readers who have sent in errata. A partial list of readers who sent in errata is: Josh Abrams, Lauren Austin, Vo Duy Cuong, Nicholas Devin, Michael Dyrud, Gavin Ferguson, Justin Garber, Charlie Jost, Jean-Christophe Langlois, Francois LeBlanc, Asher Levy, Beining Liu, Lenny Marchese, Allan Quyang, Wolfram Poh, Aaron Shotkin, John Tomkiewicz, Vincent Hew Sin Yap.



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# Calculus Notes

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The exam will not test deep calculus. You will be expected to know how to differentiate and integrate polynomials, simple rational functions, logarithms, and exponentials, but trigonometric functions and integrals will rarely appear. Fancy integration techniques are not needed.

We will review logarithmic differentiation and basic integration techniques.

**Logarithmic differentiation** When a function  $f(x)$  is a product or quotient of several functions, and you need to evaluate its derivative at a point, it is often easier to use logarithmic differentiation rather than to differentiate the function directly. Logarithmic differentiation is based on the formula for the derivative of a logarithm:

$$\frac{d \ln f(x)}{dx} = \frac{df(x)/dx}{f(x)}$$

It follows that

$$\frac{df(x)}{dx} = f(x) \left( \frac{d \ln f(x)}{dx} \right)$$

In other words, the derivative of a function is the function times the derivative of its logarithm. If the function is a product or quotient of functions, its derivative will be a sum or difference of logarithms of those functions, and that will be easier to differentiate.

**Partial fraction decomposition** You should know that  $x/(1+x) = 1 - 1/(1+x)$ , for example, so that

$$\int_0^3 \frac{x \, dx}{1+x} = \int_0^3 \left( 1 - \frac{1}{1+x} \right) dx = 3 - \ln(1+x) \Big|_0^3 = 3 - \ln 4$$

Any partial fraction decomposition more complicated is unlikely to appear.

This particular integral can also be evaluated using substitution, as we will now discuss.

**Substitution** There are two types of substitution.

The first type assists you to identify an antiderivative. Suppose you have

$$\int_0^3 x e^{-x^2/2} \, dx$$

You may realize that the integrand is negative the derivative of  $e^{-x^2/2}$ . When you differentiate  $e^{-x^2/2}$ , by the chain rule, you multiply by the derivative of  $-x^2/2$ , which is  $-x$ , and obtain  $-x e^{-x^2/2}$ . So the integral is  $-e^{-x^2/2}$  evaluated at 3 minus the same evaluated at 0. But if you didn't recognize this integrand, you could substitute  $y = -x^2/2$  and get:

$$\begin{aligned} dy &= -x \, dx \\ x = 0 &\Rightarrow y = 0 \\ x = 3 &\Rightarrow y = -4.5 \\ \int_0^3 x e^{-x^2/2} \, dx &= \int_0^{-4.5} -e^y \, dy \\ &= -e^y \Big|_0^{-4.5} \\ &= 1 - e^{-4.5} \end{aligned}$$

The second type simplifies an integral or makes it doable. We evaluated  $\int_0^3 \frac{x dx}{1+x}$  above using partial fraction decomposition. An alternative would be to set  $y = 1 + x$ ,  $dy = dx$ . The bounds of the integral become 1 and 4, since  $0 + 1 = 1$  and  $3 + 1 = 4$ , and we get

$$\begin{aligned}\int_0^3 \frac{x dx}{1+x} &= \int_1^4 \frac{(y-1) dy}{y} \\ &= \int_1^4 \left(1 - \frac{1}{y}\right) dy \\ &= (y - \ln y) \Big|_1^4 \\ &= (4 - 1) + (\ln 4 - \ln 1) = 3 - \ln 4\end{aligned}$$

As another example, suppose you have

$$\int_0^1 x(1-x)^8 dx$$

You could expand  $(1-x)^8$ , multiply it by  $x$ , and integrate 9 terms. But it is easier to substitute  $y = 1 - x$  and get

$$\begin{aligned}dy &= -dx \\ x = 0 &\Rightarrow y = 1 \\ x = 1 &\Rightarrow y = 0 \\ \int_0^1 x(1-x)^8 dx &= \int_1^0 -(1-y)y^8 dy \\ &= \int_0^1 (y^8 - y^9) dy \\ &= \frac{1}{9} - \frac{1}{10} = \frac{1}{90}\end{aligned}$$

Another technique to evaluate this integral is integration by parts, differentiating  $x$  and integrating  $(1-x)^8$ .

**Integration by parts** When integrating by parts, we use  $\int u dv = uv - \int v du$ . We integrate one expression and evaluate its product with the other, then differentiate the other and evaluate the integral with the derivative and the integrated expression.

If you are given

$$\int_0^1 x(1-x)^8 dx$$

set  $u = x$  and  $dv = (1-x)^8 dx$  and you get

$$\begin{aligned}\int_0^1 x(1-x)^8 dx &= -\frac{x(1-x)^9}{9} \Big|_0^1 + \int_0^1 \frac{(1-x)^9}{9} dx \\ &= -\frac{(1-x)^{10}}{90} \Big|_0^1 = \frac{1}{90}\end{aligned}$$

A technique that is helpful, especially if integration by parts has to be repeated, is tabular integration. Set up a table. Each row has two columns. The first row has the two expressions, the one you want to differentiate (left column) and the one you want to integrate (right column). On each successive row, differentiate the left column entry from the previous row and integrate the right column entry from the previous row. Keep doing this until the left column entry is 0. Then the antiderivative is the alternating sum of the products of the  $k^{\text{th}}$  entry of the left column and the  $k + 1^{\text{st}}$  entry of the second column,  $k = 1, 2, \dots$  up to but not including the final 0 in the left column. The sign of each summand is  $(-1)^{k-1}$ .

For example, suppose you want to evaluate

$$\int_0^3 x^2 e^{x/5} dx$$

The table looks like this:

+	$x^2$	—	$e^{x/5}$
-	$2x$	—	$5e^{x/5}$
+	$2$	—	$25e^{x/5}$
	$0$	—	$125e^{x/5}$

The entries connected with a line are multiplied, and added or subtracted as indicated by the sign. The result is

$$\int_0^3 x^2 e^{x/5} dx = \left( 5x^2 e^{x/5} - 50x e^{x/5} + 250e^{x/5} \right) \Big|_0^3 = 14.20723$$

We often integrate  $x e^{-ax}$ . If you wish, you may memorize the following results, especially the first one, so that you don't have to integrate by parts each time you need them:

$$\int_0^{\infty} x e^{-ax} dx = \frac{1}{a^2} \quad \text{for } a > 0 \quad (1)$$

$$\int_0^{\infty} x^2 e^{-ax} dx = \frac{2}{a^3} \quad \text{for } a > 0 \quad (2)$$

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# Lesson 1

## Sets

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Most things in life are not certain. Probability is a mathematical model for uncertain events. Probability assigns a number between 0 and 1 to each event. This number may have the following meanings:

1. It may indicate that of all the events in the universe, the proportion of them included in this event is that number. For example, if one says that 70% of the population owns a car, it means that the number of people owning a car is 70% of the number of people in the population.
2. It may indicate that in the long run, this event will occur that proportion of the time. For example, if we say that a certain medicine cures an illness 80% of the time, it means that we expect that if we have a large number of people, let's say 1000, with that illness who take the medicine, approximately 800 will be cured.

From a mathematical viewpoint, probability is a function from the space of events to the interval of real numbers between 0 and 1. We write this function as  $P[A]$ , where  $A$  is an event. We often want to study combinations of events. For example, if we are studying people, events may be "male", "female", "married", and "single". But we may also want to consider the event "young and married", or "male or single". To understand how to manipulate combinations of events, let's briefly study set theory. An event can be treated as a set.

A set is a collection of objects. The objects in the set are called members of the set. Two special sets are

1. The entire space. I'll use  $\Omega$  for the entire space, but there is no standard notation. All members of all sets must come from  $\Omega$ .
2. The empty set, usually denoted by  $\emptyset$ . This set has no members.

There are three important operations on sets:

**Union** If  $A$  and  $B$  are sets, we write the union as  $A \cup B$ . It is defined as the set whose members are all the members of  $A$  plus all the members of  $B$ . Thus if  $x$  is in  $A \cup B$ , then either  $x$  is in  $A$  or  $x$  is in  $B$ .  $x$  may be a member of both  $A$  and  $B$ . The union of two sets is always at least as large as each of the two component sets.

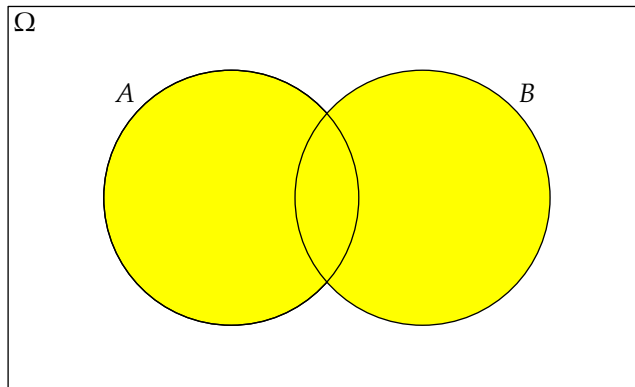
**Intersection** If  $A$  and  $B$  are sets, we write the intersection as  $A \cap B$ . It is defined as the set whose members are in both  $A$  and  $B$ . The intersection of two sets is always no larger than each of the two component sets.

**Complement** If  $A$  is a set, its complement is the set of members of  $\Omega$  that are not members of  $A$ . There is no standard notation for complement; different textbooks use  $A'$ ,  $A^c$ , and  $\bar{A}$ . I'll use  $A'$ , the notation used in SOA sample questions. Interestingly, SOA sample solutions use  $A^c$  instead.

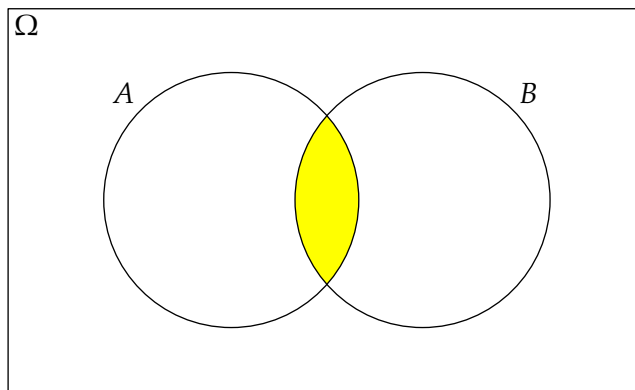
*Venn diagrams* are used to portray sets and their relationships. Venn diagrams display a set as a closed figure, usually a circle or an ellipse, and different sets are shown as intersecting if they have common elements. We present three Venn diagrams here, each showing a function of two sets as a shaded region. Figure 1.1 shows the union of two sets,  $A$  and  $B$ . Figure 1.2 shows the intersection of  $A$  and  $B$ . Figure 1.3 shows the complement of  $A \cup B$ . In these diagrams,  $A$  and  $B$  have a non-trivial intersection. However, if  $A$  and  $B$  are two sets with no intersection, we say that  $A$  and  $B$  are *mutually exclusive*. In symbols, mutually exclusive means  $A \cap B = \emptyset$ .

Important set properties are:

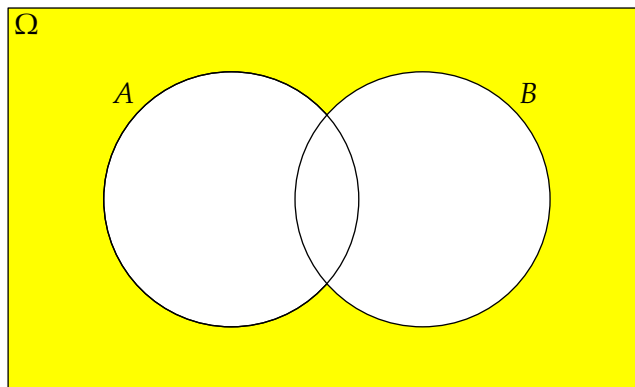
1. Associative property:  $(A \cup B) \cup C = A \cup (B \cup C)$  and  $(A \cap B) \cap C = A \cap (B \cap C)$
2. Distributive property:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  and  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. Distributive property for complement:  $(A \cup B)' = A' \cap B'$  and  $(A \cap B)' = A' \cup B'$



**Figure 1.1:**  $A \cup B$



**Figure 1.2:**  $A \cap B$



**Figure 1.3:**  $(A \cup B)'$

**EXAMPLE 1A** Simplify  $(A \cup B) \cap (A \cup B')$ .

**SOLUTION:** By the distributive property,

$$(A \cup B) \cap (A \cup B') = A \cup (B \cap B')$$

But  $B$  and  $B'$  are mutually exclusive:  $B \cap B' = \emptyset$ . So

$$(A \cup B) \cap (A \cup B') = A \cup \emptyset = \boxed{A}$$

□

Probability theory has three axioms:

1. The probability of any set is greater than or equal to 0.
2. The probability of the entire space is 1.
3. The probability of a countable union of mutually exclusive sets is the sum of the probabilities of the sets.

From these axioms, many properties follow, such as:

1.  $P[A] \leq 1$  for any  $A$ .
2.  $P[A'] = 1 - P[A]$ .
3.  $P[A \cap B] \leq P[A]$ .

Looking at Figure 1.1, we see that  $A \cup B$  has three mutually exclusive components:  $A \cap B'$ ,  $B \cap A'$ , and the intersection of the two sets  $A \cap B$ . To compute  $P[A \cup B]$ , if we add together  $P[A]$  and  $P[B]$ , we double count the intersection, so we must subtract its probability. Thus

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \quad (1.1)$$

This can also be expressed with  $\cup$  and  $\cap$  reversed:

$$P[A \cap B] = P[A] + P[B] - P[A \cup B] \quad (1.2)$$

**EXAMPLE 1B** A company is trying to plan social activities for its employees. It finds:

- i) 35% of employees do not attend the company picnic.
- ii) 80% of employees do not attend the golf and tennis outing.
- iii) 25% of employees do not attend the company picnic and also don't attend the golf and tennis outing.

What percentage of employees attend both the company picnic and the golf and tennis outing?

**SOLUTION:** Let  $A$  be the event of attending the picnic and  $B$  the event of attending the golf and tennis outing. Then

$$P[A \cap B] = P[A] + P[B] - P[A \cup B]$$

$$P[A] = 1 - P[A'] = 1 - 0.35 = 0.65$$

$$P[B] = 1 - P[B'] = 1 - 0.80 = 0.20$$

$$P[A \cup B] = 1 - P[(A \cup B)'] = 1 - P[A' \cap B'] = 1 - 0.25 = 0.75$$

$$P[A \cap B] = 0.65 + 0.20 - 0.75 = \boxed{0.10}$$

□

Equations (1.1) and (1.2) are special cases of *inclusion-exclusion equations*. The generalization deals with unions or intersections of any number of sets. For the probability of the union of  $n$  sets, add up the probabilities of the sets, then subtract the probabilities of intersections of 2 sets, add the probabilities of intersections of 3 sets, and so on, until you get to  $n$ :

$$P\left[\bigcup_{i=1}^n A_i\right] = \sum_{i=1}^n P[A_i] - \sum_{i \neq j} P[A_i \cap A_j] + \sum_{i \neq j \neq k} P[A_i \cap A_j \cap A_k] - \dots + (-1)^{n-1} P[A_1 \cap A_2 \cap \dots \cap A_n] \quad (1.3)$$

**Table 1.1:** Formula summary for probabilities of sets

<p><b>Set properties</b></p> $(A \cup B) \cup C = A \cup (B \cup C) \text{ and } (A \cap B) \cap C = A \cap (B \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \text{ and } A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $(A \cup B)' = A' \cap B' \text{ and } (A \cap B)' = A' \cup B'$ <p><b>Mutually exclusive:</b> <math>A \cap B = \emptyset</math>  <math>P[A \cup B] = P[A] + P[B]</math> for <math>A</math> and <math>B</math> mutually exclusive</p> <p><b>Inclusion-exclusion equations:</b></p> $P[A \cup B] = P[A] + P[B] - P[A \cap B] \tag{1.1}$ $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$
--

On an exam, it is unlikely you would need this formula for more than 3 sets. With 3 sets, there are probabilities of three intersections of two sets to subtract and one intersection of all three sets to add:

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$$

**EXAMPLE 1C** Your company is trying to sell additional policies to group policyholders. It finds:

- i) 10% of customers do not have group life, group health, or group disability.
- ii) 25% of customers have group life.
- iii) 75% of customers have group health.
- iv) 20% of customers have group disability.
- v) 40% of customers have group life and group health.
- vi) 22% of customers have group disability and group health.
- vii) 5% of customers have group life and group disability.

Calculate the percentage of customers who have all three coverages: group life, group health, and group disability.

**SOLUTION:** Each insurance coverage is an event, and we are given intersections of events, so we'll use inclusion-exclusion on the union. The first statement implies that the probability of the union of all three events is  $1 - 0.1 = 0.9$ . Let the probability of the intersection, which is what we are asked for, be  $x$ . Then

$$0.9 = 0.25 + 0.75 + 0.20 - 0.40 - 0.22 - 0.05 + x = 0.53 + x$$

It follows that  $x = 0.37$ . □

Most exam questions based on this lesson will require use of the inclusion-exclusion equations for 2 or 3 sets.

## Exercises

- 1.1.** [110-S83:17] If  $P[X] = 0.25$  and  $P[Y] = 0.80$ , then which of the following inequalities must be true?
- I.  $P[X \cap Y] \leq 0.25$   
II.  $P[X \cap Y] \geq 0.20$   
III.  $P[X \cap Y] \geq 0.05$
- (A) I only                      (B) I and II only                      (C) I and III only                      (D) II and III only  
(E) The correct answer is not given by (A), (B), (C), or (D).
- 1.2.** [110-S85:29] Let  $E$  and  $F$  be events such that  $P[E] = \frac{1}{2}$ ,  $P[F] = \frac{1}{2}$ , and  $P[E' \cap F'] = \frac{1}{3}$ . Then  $P[E \cup F] =$
- (A)  $\frac{1}{4}$                       (B)  $\frac{2}{3}$                       (C)  $\frac{3}{4}$                       (D)  $\frac{5}{6}$                       (E) 1
- 1.3.** [110-S88:10] If  $E$  and  $F$  are events for which  $P[E \cup F] = 1$ , then  $P[E' \cup F']$  must equal
- (A) 0  
(B)  $P[E'] + P[F'] - P[E']P[F']$   
(C)  $P[E'] + P[F']$   
(D)  $P[E'] + P[F'] - 1$   
(E) 1
- 1.4.** [110-W96:23] Let  $A$  and  $B$  be events such that  $P[A] = 0.7$  and  $P[B] = 0.9$ . Calculate the largest possible value of  $P[A \cup B] - P[A \cap B]$ .
- (A) 0.20                      (B) 0.34                      (C) 0.40                      (D) 0.60                      (E) 1.60
- 1.5.** You are given that  $P[A \cup B] - P[A \cap B] = 0.3$ ,  $P[A] = 0.8$ , and  $P[B] = 0.7$ . Determine  $P[A \cup B]$ .
- 1.6.** [S01:12, Sample:3] You are given  $P[A \cup B] = 0.7$  and  $P[A \cup B'] = 0.9$ . Calculate  $P[A]$ .
- (A) 0.2                      (B) 0.3                      (C) 0.4                      (D) 0.6                      (E) 0.8
- 1.7.** [1999 Sample:1] A marketing survey indicates that 60% of the population owns an automobile, 30% owns a house, and 20% owns both an automobile and a house. Calculate the probability that a person chosen at random owns an automobile or a house, but not both.
- (A) 0.4                      (B) 0.5                      (C) 0.6                      (D) 0.7                      (E) 0.9



**1.8.** [S03:1,Sample:1] A survey of a group's viewing habits over the last year revealed the following information:

- i) 28% watched gymnastics
- ii) 29% watched baseball
- iii) 19% watched soccer
- iv) 14% watched gymnastics and baseball
- v) 12% watched baseball and soccer
- vi) 10% watched gymnastics and soccer
- vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

- (A) 24                      (B) 36                      (C) 41                      (D) 52                      (E) 60

**1.9.** A survey of a group's viewing habits over the last year revealed the following information:

- i) 28% watched gymnastics
- ii) 29% watched baseball
- iii) 19% watched soccer
- iv) 14% watched gymnastics and baseball
- v) 12% watched baseball and soccer
- vi) 10% watched gymnastics and soccer
- vii) 8% watched all three sports.

Calculate the percentage of the group that watched baseball but neither soccer nor gymnastics during the last year.

**1.10.** An insurance company finds that among its policyholders:

- i) Each one has either health, dental, or life insurance.
- ii) 81% have health insurance.
- iii) 36% have dental insurance.
- iv) 24% have life insurance.
- v) 5% have all three insurance coverages.
- vi) 14% have dental and life insurance.
- vii) 12% have health and life insurance.

Determine the percentage of policyholders having health insurance but not dental insurance.

**1.11.** [Sample:243] An insurance agent's files reveal the following facts about his policyholders:

- i) 243 own auto insurance.
- ii) 207 own homeowner insurance.
- iii) 55 own life insurance and homeowner insurance.
- iv) 96 own auto insurance and homeowner insurance.
- v) 32 own life insurance, auto insurance and homeowner insurance.
- vi) 76 more clients own only auto insurance than only life insurance.
- vii) 270 own only one of these three insurance products.

Calculate the total number of the agent's policyholders who own at least one of these three insurance products.

- (A) 389                      (B) 407                      (C) 423                      (D) 448                      (E) 483

**1.12.** [Sample:244] A profile of the investments owned by an agent's clients follows:

- i) 228 own annuities.
- ii) 220 own mutual funds.
- iii) 98 own life insurance and mutual funds.
- iv) 93 own annuities and mutual funds.
- v) 16 own annuities, mutual funds, and life insurance.
- vi) 45 more clients own only life insurance than own only annuities.
- vii) 290 own only one type of investment (i.e., annuity, mutual fund, or life insurance).

Calculate the agent's total number of clients.

- (A) 455                      (B) 495                      (C) 496                      (D) 500                      (E) 516

**1.13.** [S00:1,Sample:2] The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work.

Calculate the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

- (A) 0.05                      (B) 0.12                      (C) 0.18                      (D) 0.25                      (E) 0.35

**1.14.** In a certain town, there are 1000 cars. All cars are white, blue, or gray, and are either sedans or SUVs. There are 300 white cars, 400 blue cars, 760 sedans, 180 white sedans, and 320 blue sedans.

Determine the number of gray SUVs.

**1.15.** [F00:3,Sample:5] An auto insurance company has 10,000 policyholders. Each policyholder is classified as

- i) young or old;
- ii) male or female; and
- iii) married or single

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males.

Calculate the number of the company's policyholders who are young, female, and single.

- (A) 280                      (B) 423                      (C) 486                      (D) 880                      (E) 896

**1.16.** An auto insurance company has 10,000 policyholders. Each policyholder is classified as

- i) young or old;
- ii) male or female; and
- iii) married or single

Of these policyholders, 4000 are young, 5600 are male, and 3500 are married. The policyholders can also be classified as 2820 young males, 1540 married males, and 1300 young married persons. Finally, 670 of the policyholders are young married males.

How many of the company's policyholders are old, female, and single?

**1.17.** [F01:9,Sample:8] Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist.

Calculate the probability that a randomly chosen member of this group visits a physical therapist.

- (A) 0.26                      (B) 0.38                      (C) 0.40                      (D) 0.48                      (E) 0.62

**1.18.** For new hires in an actuarial student program:

- i) 20% have a postgraduate degree.
- ii) 30% are Associates.
- iii) 60% have 2 or more years of experience.
- iv) 14% have both a postgraduate degree and are Associates.
- v) The proportion who are Associates and have 2 or more years of experience is twice the proportion who have a postgraduate degree and have 2 or more years experience.
- vi) 25% do not have a postgraduate degree, are not Associates, and have less than 2 years of experience.
- vii) Of those who are Associates and have 2 or more years experience, 10% have a postgraduate degree.

Calculate the percentage that have a postgraduate degree, are Associates, and have 2 or more years experience.

**1.19.** [Sample:246] An actuary compiles the following information from a portfolio of 1000 homeowners insurance policies:

- i) 130 policies insure three-bedroom homes.
- ii) 280 policies insure one-story homes.
- iii) 150 policies insure two-bath homes.
- iv) 30 policies insure three-bedroom, two-bath homes.
- v) 50 policies insure one-story, two-bath homes.
- vi) 40 policies insure three-bedroom, one-story homes.
- vii) 10 policies insure three-bedroom, one-story, two-bath homes.

Calculate the number of homeowners policies in the portfolio that insure neither one-story nor two-bath nor three-bedroom homes.

- (A) 310                      (B) 450                      (C) 530                      (D) 550                      (E) 570

**1.20.** [S03:5,Sample:9] An insurance company examines its pool of auto insurance customers and gathers the following information:

- i) All customers insure at least one car.
- ii) 70% of the customers insure more than one car.
- iii) 20% of the customers insure a sports car.
- iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

- (A) 0.13                      (B) 0.21                      (C) 0.24                      (D) 0.25                      (E) 0.30

**1.21.** An employer offers employees the following coverages:

- i) Vision insurance
- ii) Dental insurance
- iii) Long term care (LTC) insurance

Employees who enroll for insurance must enroll for at least two coverages. You are given

- i) The probability of enrolling for vision insurance is 40%.
- ii) The probability of enrolling for dental insurance is 80%.
- iii) The probability of enrolling for LTC insurance is 70%.
- iv) The probability of enrolling for all three insurances is 20%.

Calculate the probability of not enrolling for any insurance.

**1.22. [Sample:126]** Under an insurance policy, a maximum of five claims may be filed per year by a policyholder. Let  $p(n)$  be the probability that a policyholder files  $n$  claims during a given year, where  $n = 0, 1, 2, 3, 4, 5$ . An actuary makes the following observations:

- i)  $p(n) \geq p(n + 1)$  for  $0, 1, 2, 3, 4$ .
- ii) The difference between  $p(n)$  and  $p(n + 1)$  is the same for  $n = 0, 1, 2, 3, 4$ .
- iii) Exactly 40% of policyholders file fewer than two claims during a given year.

Calculate the probability that a random policyholder will file more than three claims during a given year.

- (A) 0.14                      (B) 0.16                      (C) 0.27                      (D) 0.29                      (E) 0.33

**1.23. [Sample:128]** An insurance agent offers his clients auto insurance, homeowners insurance and renters insurance. The purchase of homeowners insurance and the purchase of renters insurance are mutually exclusive. The profile of the agent's clients is as follows:

- i) 17% of the clients have none of these three products.
- ii) 64% of the clients have auto insurance.
- iii) Twice as many of the clients have homeowners insurance as have renters insurance.
- iv) 35% of the clients have two of these three products.
- v) 11% of the clients have homeowners insurance, but not auto insurance.

Calculate the percentage of the agent's clients that have both auto and renters insurance.

- (A) 7%                      (B) 10%                      (C) 16%                      (D) 25%                      (E) 28%

**1.24. [Sample:134]** A mattress store sells only king, queen and twin-size mattresses. Sales records at the store indicate that one-fourth as many queen-size mattresses are sold as king and twin-size mattresses combined. Records also indicate that three times as many king-size mattresses are sold as twin-size mattresses.

Calculate the probability that the next mattress sold is either king or queen-size.

- (A) 0.12                      (B) 0.15                      (C) 0.80                      (D) 0.85                      (E) 0.95

**1.25. [Sample:143]** The probability that a member of a certain class of homeowners with liability and property coverage will file a liability claim is 0.04, and the probability that a member of this class will file a property claim is 0.10. The probability that a member of this class will file a liability claim but not a property claim is 0.01.

Calculate the probability that a randomly selected member of this class of homeowners will not file a claim of either type.

- (A) 0.850                      (B) 0.860                      (C) 0.864                      (D) 0.870                      (E) 0.890

**1.26. [Sample:146]** A survey of 100 TV watchers revealed that over the last year:

- i) 34 watched CBS.
- ii) 15 watched NBC.
- iii) 10 watched ABC.
- iv) 7 watched CBS and NBC.
- v) 6 watched CBS and ABC.
- vi) 5 watched NBC and ABC.
- vii) 4 watched CBS, NBC, and ABC.
- viii) 18 watched HGTV and of these, none watched CBS, NBC, or ABC.

Calculate how many of the 100 TV watchers did not watch any of the four channels (CBS, NBC, ABC or HGTV).

- (A) 1                      (B) 37                      (C) 45                      (D) 55                      (E) 82

**1.27.** [Sample:179] This year, a medical insurance policyholder has probability 0.70 of having no emergency room visits, 0.85 of having no hospital stays, and 0.61 of having neither emergency room visits nor hospital stays.

Calculate the probability that the policyholder has at least one emergency room visit and at least one hospital stay this year.

- (A) 0.045                      (B) 0.060                      (C) 0.390                      (D) 0.667                      (E) 0.840

**1.28.** [Sample:198] In a certain group of cancer patients, each patient's cancer is classified in exactly one of the following five stages: stage 0, stage 1, stage 2, stage 3, or stage 4.

- i) 75% of the patients in the group have stage 2 or lower.
- ii) 80% of the patients in the group have stage 1 or higher.
- iii) 80% of the patients in the group have stage 0, 1, 3, or 4.

One patient from the group is randomly selected.

Calculate the probability that the selected patient's cancer is stage 1.

- (A) 0.20                      (B) 0.25                      (C) 0.35                      (D) 0.48                      (E) 0.65

**1.29.** [Sample:207] A policyholder purchases automobile insurance for two years. Define the following events:

F = the policyholder has exactly one accident in year one.

G = the policyholder has one or more accidents in year two.

Define the following events:

- i) The policyholder has exactly one accident in year one and has more than one accident in year two.
- ii) The policyholder has at least two accidents during the two-year period.
- iii) The policyholder has exactly one accident in year one and has at least one accident in year two.
- iv) The policyholder has exactly one accident in year one and has a total of two or more accidents in the two-year period.
- v) The policyholder has exactly one accident in year one and has more accidents in year two than in year one.

Determine the number of events from the above list of five that are the same as  $F \cap G$ .

- (A) None
- (B) Exactly one
- (C) Exactly two
- (D) Exactly three
- (E) All

**1.30.** [F01:1,Sample:4] An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44.

Calculate the number of blue balls in the second urn.

- (A) 4                      (B) 20                      (C) 24                      (D) 44                      (E) 64

**1.31.** [Sample:254] The annual numbers of thefts a homeowners insurance policyholder experiences are analyzed over three years. Define the following events:

- i) A = the event that the policyholder experiences no thefts in the three years.
- ii) B = the event that the policyholder experiences at least one theft in the second year.
- iii) C = the event that the policyholder experiences exactly one theft in the first year.
- iv) D = the event that the policyholder experiences no thefts in the third year.
- v) E = the event that the policyholder experiences no thefts in the second year, and at least one theft in the third year.

Determine which three events satisfy the condition that the probability of their union equals the sum of their probabilities.

- (A) Events A, B, and E
- (B) Events A, C, and E
- (C) Events A, D, and E
- (D) Events B, C, and D
- (E) Events B, C, and E

**1.32.** [S01:31, Sample:15] An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are  $\frac{1}{3}$ ,  $\frac{1}{4}$ , and  $\frac{5}{12}$ , respectively.

Determine the probability that a randomly chosen employee will choose no supplementary coverage.

- (A) 0
- (B)  $\frac{47}{144}$
- (C)  $\frac{1}{2}$
- (D)  $\frac{97}{144}$
- (E)  $\frac{7}{9}$

**1.33.** [Sample:255] Four letters to different insureds are prepared along with accompanying envelopes. The letters are put into the envelopes randomly.

Calculate the probability that at least one letter ends up in its accompanying envelope.

- (A)  $27/256$
- (B)  $1/4$
- (C)  $11/24$
- (D)  $5/8$
- (E)  $3/4$

**1.34.** The probability of event  $U$  is 0.8 and the probability of event  $V$  is 0.4.

What is the lowest possible probability of the event  $U \cap V$ ?

## Solutions

**1.1.** Since  $X \cap Y \subset X$ ,  $P[X \cap Y] \leq P[X] = 0.25$ . And since  $P[X \cup Y] \leq 1$  and  $P[X \cup Y] = P[X] + P[Y] - P[X \cap Y]$ , it follows that

$$\begin{aligned} P[X] + P[Y] - P[X \cap Y] &\leq 1 \\ 0.25 + 0.80 - P[X \cap Y] &\leq 1 \end{aligned}$$

so  $P[X \cap Y] \geq 0.05$ . One can build a counterexample to II by arranging for the union of  $X$  and  $Y$  to equal the entire space. (C)

**1.2.** Split  $E \cup F'$  into the following two disjoint sets:  $E$  and  $E' \cap F'$ . These two sets are disjoint since  $E \cap E' = \emptyset$ , and comprise  $E \cup F'$  because everything in  $E$  is included and anything in  $F'$  is either in  $E$  or is in  $E' \cap F'$ .

$$P[E \cup F'] = P[E] + P[E' \cap F'] = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \quad \text{(D)}$$

1.3.

$$\begin{aligned}
 P[E' \cup F'] &= P[E'] + P[F'] - P[E' \cap F'] \\
 E' \cap F' &= (E \cup F)' \\
 P[E' \cap F'] &= 1 - P[E \cup F] = 0 \\
 P[E' \cup F'] &= P[E'] + P[F'] \quad \text{(C)}
 \end{aligned}$$

1.4.  $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ . From this equation, we see that for fixed  $A$  and  $B$ , the smaller  $P[A \cap B]$  is, the larger  $P[A \cup B]$  is. Therefore, maximizing  $P[A \cup B]$  also maximizes  $P[A \cup B] - P[A \cap B]$ . The highest possible value of  $P[A \cup B]$  is 1. Then

$$P[A \cap B] = P[A] + P[B] - P[A \cup B] = 0.7 + 0.9 - 1 = 0.6$$

and

$$P[A \cup B] - P[A \cap B] = 1 - 0.6 = \mathbf{0.4} \quad \text{(C)}$$

1.5.  $P[A \cap B] = P[A] + P[B] - P[A \cup B]$ , so  $P[A \cap B] = 0.8 + 0.7 - P[A \cup B] = 1.5 - P[A \cup B]$ . Then substituting into the first probability that we are given,  $2P[A \cup B] - 1.5 = 0.3$ , so  $P[A \cup B] = \mathbf{0.9}$ .

1.6. The union of  $A \cup B$  and  $A \cup B'$  is the entire space, since  $B \cup B' = \Omega$ , the entire space. The probability of the union is 1. By the inclusion-exclusion principle

$$P[(A \cup B) \cap (A \cup B')] = 0.7 + 0.9 - 1 = 0.6$$

Now,

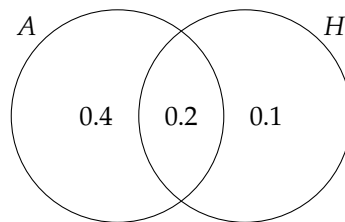
$$(A \cup B) \cap (A \cup B') = A \cap (B \cup B') = A$$

so  $P[A] = \mathbf{0.6}$  (D)

1.7. If we let  $A$  be the set of automobile owners and  $H$  the set of house owners, we want

$$(P[A] - P[A \cap H]) + (P[H] - P[A \cap H]) = P[A] + P[H] - 2P[A \cap H]$$

Using what we are given, this equals  $0.6 + 0.3 - 2(0.2) = \mathbf{0.5}$ . (B)



1.8. If the sets of watching gymnastics, baseball, and soccer are  $G$ ,  $B$  and  $S$  respectively, we want  $G' \cap B' \cap S' = (G \cup B \cup S)'$ , and

$$\begin{aligned}
 P[G \cup B \cup S] &= P[G] + P[B] + P[S] - P[G \cap B] - P[G \cap S] - P[B \cap S] + P[G \cap B \cap S] \\
 &= 0.28 + 0.29 + 0.19 - 0.14 - 0.12 - 0.10 + 0.08 = 0.48
 \end{aligned}$$

So the answer is  $1 - 0.48 = \mathbf{0.52}$ . (D)

1.9. The percentage watching baseball is 29%. Of these, 14% also watched gymnastics and 12% also watched soccer, so we subtract these. However, in this subtraction we have double counted those who watch all three sports (8%), so we add that back in. The answer is  $29\% - 14\% - 12\% + 8\% = \mathbf{11\%}$ .

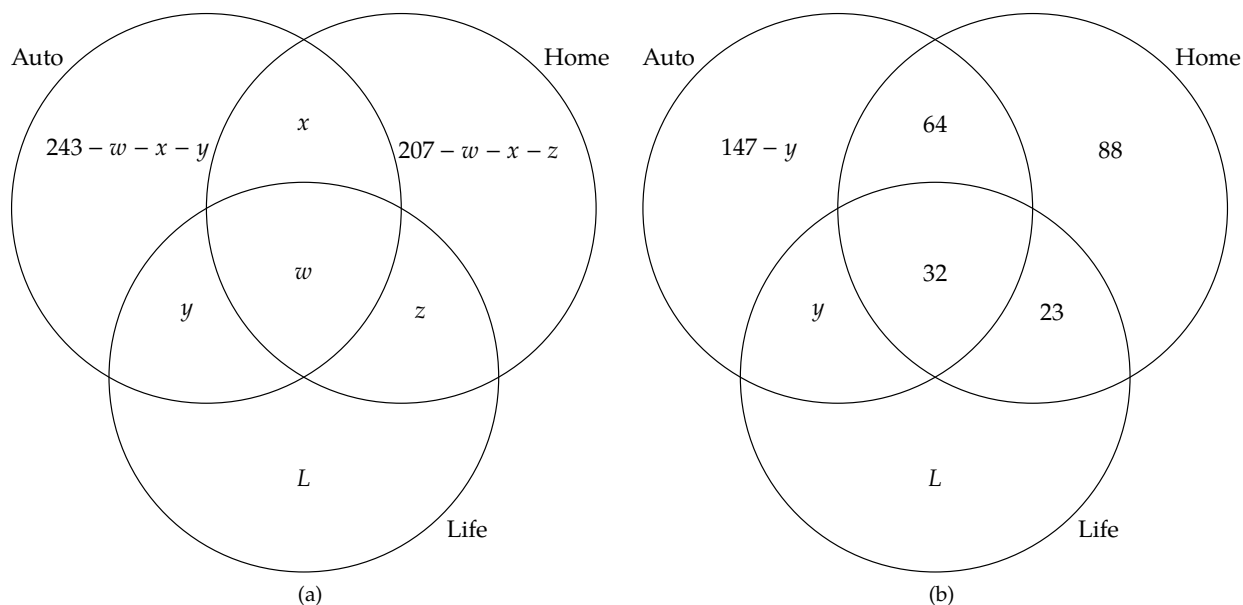


Figure 1.4: Venn diagrams for exercise 1.11

**1.10.** Since everyone has insurance, the union of the three insurances has probability 1. If we let  $H$ ,  $D$ , and  $L$  be health, dental, and life insurance respectively, then

$$\begin{aligned}
 1 &= P[H \cup D \cup L] \\
 &= P[H] + P[D] + P[L] - P[H \cap D] - P[H \cap L] - P[D \cap L] + P[H \cap D \cap L] \\
 &= 0.81 + 0.36 + 0.24 - P[H \cap D] - 0.14 - 0.12 + 0.05 \\
 &= 1.20 - P[H \cap D]
 \end{aligned}$$

so  $P[H \cap D] = 0.20$ . Since 81% have health insurance, this implies that  $0.81 - 0.20 = \boxed{61\%}$  have health insurance but not dental insurance.

**1.11.** A Venn diagram is helpful here. Figure 1.4a incorporates i) and ii). Based on iii), iv), and v),  $w + z = 55$ ,  $w + x = 96$ , and  $w = 32$ , resulting in Figure 1.4b. Based on vi),  $147 - y = L + 76$ , so  $L = 71 - y$ , as in Figure 1.5a. Now we use vii) to solve for  $y$  and  $L$ :

$$\begin{aligned}
 147 - y + 88 + 71 - y &= 270 \\
 306 - 2y &= 270 \\
 y &= 18
 \end{aligned}$$

Now we have Figure 1.5b. We add up all the numbers to get

$$129 + 64 + 88 + 18 + 32 + 23 + 53 = \boxed{407} \quad (\mathbf{B})$$

**1.12.** Virtually identical to the previous exercise, and you can use the same Venn diagrams with updated numbers. We have  $98 - 16 = 82$  with only life insurance and mutual funds and  $93 - 16 = 77$  with only annuities and mutual funds. The number with mutual funds only is  $220 - 77 - 16 - 82 = 45$ . Let  $y$  be the number with annuities and life insurance only, and  $L$  the number with life insurance only. The number with annuities only is  $228 - 93 - y = 135 - y$ . Then from vi),

$$135 - y = L - y - 45$$



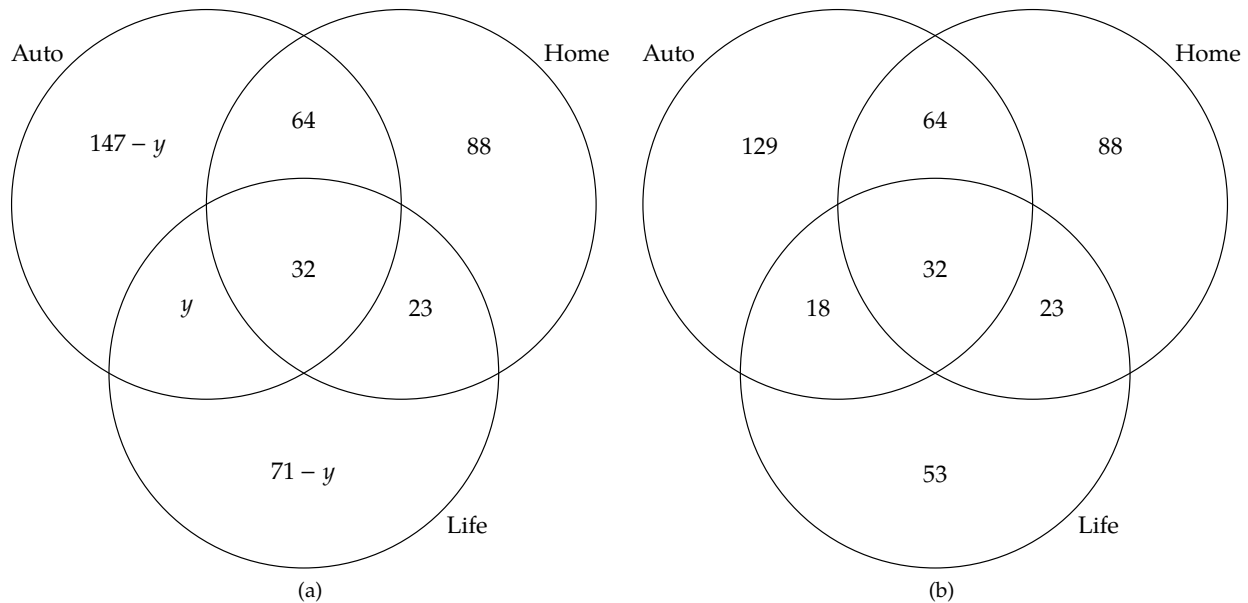


Figure 1.5: Venn diagrams for exercise 1.11

$$L = 180 - y$$

The number From vii),  $135 - y + 45 + 180 - y = 290$  so  $y = 35$ . The number with annuities only is  $135 - y = 100$ . The number with life insurance only is  $180 - y = 145$ . Adding up all the numbers,

$$100 + 45 + 145 + 77 + 35 + 82 + 16 = \mathbf{500} \quad (\mathbf{D})$$

**1.13.** If lab work is  $A$  and specialist is  $B$ , then  $P[A' \cap B'] = 0.35$ ,  $P[A] = 0.3$ , and  $P[B] = 0.4$ . We want  $P[A \cap B]$ . Now,  $P(A \cup B)' = P(A' \cap B') = 0.35$ , so  $P[A \cup B] = 0.65$ . Then

$$P[A \cap B] = P[A] + P[B] - P[A \cup B] = 0.3 + 0.4 - 0.65 = \mathbf{0.05} \quad (\mathbf{A})$$

**1.14.** There are  $1000 - 760 = 240$  SUVs. Of these, there are  $300 - 180 = 120$  white SUVs and  $400 - 320 = 80$  blue SUVs, so there must be  $240 - 120 - 80 = \mathbf{40}$  gray SUVs.

The following table may be helpful. Given numbers are in roman and derived numbers are in italics.

	Total	Sedan	SUV
Total	1000	760	<i>240</i>
White	300	180	<i>120</i>
Blue	400	320	<i>80</i>
Gray			<i>40</i>

**1.15.** There are 3000 young. Of those, remove 1320 young males and 1400 young marrieds. That removes young married males twice, so add back young married males. The result is  $3000 - 1320 - 1400 + 600 = \mathbf{880}$ . **(D)**

**1.16.** If the classifications are  $A$ ,  $B$  and  $C$  for young, male, and married respectively, we calculate (# denotes the number of members of a set.)

$$\#[A' \cap B' \cap C'] = \#[(A \cup B \cup C)']$$

$$\#[A \cup B \cup C] = \#[A] + \#[B] + \#[C] - \#[A \cap B] - \#[A \cap C] - \#[B \cap C] + \#[A \cap B \cap C]$$

$$= 4000 + 5600 + 3500 - 2820 - 1540 - 1300 + 670 = 8110$$

$$\#[A' \cap B' \cap C'] = 10,000 - 8,110 = \boxed{1,890}$$

**1.17.** Let  $A$  be physical therapist and  $B$  be chiropractor. We want  $P[A]$ . We are given that  $P[A \cap B] = 0.22$  and  $P[A' \cap B'] = 0.12$ . Also,  $P[B] = P[A] + 0.14$ . Then  $P[A \cup B] = 1 - P[A' \cap B'] = 0.88$ . So

$$P[A \cap B] = P[A] + P[B] - P[A \cup B]$$

$$0.22 = P[A] + P[A] + 0.14 - 0.88$$

$$P[A] = \boxed{0.48} \quad \text{(D)}$$

**1.18.** Let  $A$  be postgraduate degree,  $B$  Associates, and  $C$  2 or more years experience. Let  $x = P[B \cap C]$ . Then

$$P[A' \cap B' \cap C'] = 0.25$$

$$P[(A \cup B \cup C)'] = 0.25$$

$$P[A \cup B \cup C] = 0.75$$

$$0.75 = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$$

$$= 0.2 + 0.3 + 0.6 - 0.14 - 0.5x - x + 0.1x$$

$$= 0.96 - 1.4x$$

$$x = 0.15$$

The answer is  $0.1(0.15) = 0.015$ , or  $\boxed{1.5\%}$ .

**1.19.** This is a straightforward inclusion-exclusion principle question: to get total in at least one of the three classes, add those in 1, subtract those in 2, add those in 3:

$$130 + 280 + 150 - 30 - 50 - 40 + 10 = 450$$

That leaves  $\boxed{550}$  in none of the classes. **(D)**

**1.20.** Statement (iv) in conjunction with statement (ii) tells us that  $0.15(0.7) = 0.105$  insured more than one car including a sports car. Then from (iii),  $0.2 - 0.105 = 0.095$  insure one car that is a sports car. Since 0.3 insure one car, that leaves  $0.3 - 0.095 = \boxed{0.205}$  who insure one car that is not a sports car. **(B)**

**1.21.** If we let  $A, B, C$  be vision, dental, and LTC, then

$$P[A' \cap B' \cap C'] = P[(A \cup B \cup C)'] = 1 - P[A \cup B \cup C]$$

Since nobody enrolls for just one coverage,

$$P[A] = P[A \cap B] + P[A \cap C] - P[A \cap B \cap C]$$

and similar equations can be written for  $P[B]$  and  $P[C]$ . Adding the three equations up,

$$P[A \cap B] + P[A \cap C] + P[B \cap C] = \frac{P[A] + P[B] + P[C] + 3P[A \cap B \cap C]}{2}$$

$$= \frac{0.4 + 0.8 + 0.7 + 3(0.2)}{2} = 1.25$$

Using the inclusion-exclusion formula,

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$$

$$= 0.4 + 0.8 + 0.7 - 1.25 + 0.2 = 0.85$$

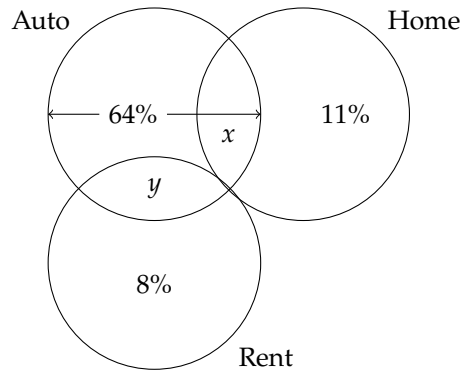
and the answer is  $1 - 0.85 = \boxed{0.15}$ .

**1.22.** Set  $d = p(0) + p(1) - p(2) - p(3)$ . We are given that  $p(0) + p(1) = 0.4$ . So  $p(2) + p(3) = 0.4 - d$ . By (ii),  $p(2) + p(3) - p(4) - p(5) = d$ . So  $p(4) + p(5) = p(2) + p(3) - d = 0.4 - 2d$ . The six probabilities  $p(0), p(1), \dots, p(5)$  must add up to 1. Therefore

$$\begin{aligned} 0.4 + (0.4 - d) + 0.4 - 2d &= 1 \\ 1.2 - 3d &= 1 \\ d &= \frac{1}{15} \end{aligned}$$

The probability of 4 or 5 claims is  $0.4 - 2/15 = \mathbf{0.266667}$ . (C)

**1.23.** A Venn diagram may be helpful here.



Let  $x$  be the proportion with both auto and home and  $y$  the proportion with both auto and renters. We are given that 17% have no product, so 83% have at least one product. 64% have auto and 11% have homeowners but not auto, so that leaves  $83\% - 64\% - 11\% = 8\%$  who have only renters insurance. Now we can set up two equations in  $x$  and  $y$ . From (iv),  $x + y = 0.35$ . From (iii),  $0.11 + x = 2(0.08 + y)$ . Solving,

$$\begin{aligned} 0.11 + (0.35 - y) &= 0.16 + 2y \\ 0.46 - 0.16 &= 3y \\ y &= \mathbf{0.10} \quad \text{(B)} \end{aligned}$$

**1.24.** We'll use  $K$ ,  $Q$ , and  $T$  for the three events king-size, queen-size, and twin-size mattresses. We're given  $P[K] + P[T] = 4P[Q]$  and  $P[K] = 3P[T]$ , and the three events are mutually exclusive and exhaustive so their probabilities must add up to 1,  $P[K] + P[T] + P[Q] = 1$ . We have three equations in three unknowns. Let's solve for  $P[T]$  and then use the complement.

$$\begin{aligned} 3P[T] + P[T] &= 4P[Q] = 4(1 - 4P[T]) \\ 4P[T] &= 4 - 16P[T] \\ P[T] &= 0.2 \end{aligned}$$

and the answer is  $1 - 0.2 = \mathbf{0.8}$ . (C)

**1.25.** We want to calculate the probability of filing some claim, or the union of those filing property and liability claims, because then we can calculate the probability of the complement of this set, those who file no claim. To calculate the probability of filing some claim, we need the probability of filing both types of claim.

The probability that a member will file a liability claim is 0.04, and of these 0.01 do not file a property claim so 0.03 do file both claims. The probability of the union of those who file liability and property claims is the sum of the probabilities of filing either type of claim, minus the probability of filing both types of claim, or  $0.04 + 0.10 - 0.03 = 0.11$ , so the probability of not filing either type of claim is  $1 - 0.11 = \mathbf{0.89}$ . (E)

**1.26.** First let's calculate how many did not watch CBS, NBC, or ABC. As usual, we add up the ones who watched one, minus the ones who watched two, plus the ones who watched three:

$$34 + 15 + 10 - 7 - 6 - 5 + 4 = 45$$

An additional 18 watched HGTV for a total of 63 who watched something.  $100 - 63 = 37$  watched nothing. **(B)**

**1.27.** Let  $A$  be the event of an emergency room visit and  $B$  the event of a hospital stay. We have  $P[A] = 1 - 0.7 = 0.3$  and  $P[B] = 1 - 0.85 = 0.15$ . Also  $P[A \cup B] = 1 - 0.61 = 0.39$ . Then the probability we want is

$$P[A \cap B] = P[A] + P[B] - P[A \cup B] = 0.3 + 0.15 - 0.39 = 0.06 \quad \text{(B)}$$

**1.28.** From ii), the probability of stage 0 is 20%. From iii), the probability of stage 2 is 20%. From i), the probability of stage 0, 1, or 2 is 75%. So the probability of stage 1 is  $0.75 - 2(0.2) = 0.35$ . **(C)**

**1.29.**  $F \cap G$  is the event of exactly one accident in year one and at least one in year two.

- i) This event is not the same since it excludes the event of one in year one and one in year two.  $\times$
- ii) This event is not the same since it includes two in year one and none in year two, among others.  $\times$
- iii) This event is the same.  $\checkmark$
- iv) This event is the same, since to have two or more accidents total, there must be one or more accidents in year two.  $\checkmark$
- v) This event is not the same since it excludes having one in year one and one in year two.  $\times$

**(C)**

**1.30.** Let  $x$  be the number of blue balls in the second urn. Then the probability that both balls are red is  $0.4(16)/(16+x)$  and the probability that both balls are blue is  $0.6x/(16+x)$ . The sum of these two expressions is 0.44, so

$$\begin{aligned} \frac{6.4 + 0.6x}{16 + x} &= 0.44 \\ 6.4 + 0.6x &= 7.04 + 0.44x \\ 0.16x &= 0.64 \\ x &= 4 \quad \text{(A)} \end{aligned}$$

**1.31.** We want to find three mutually exclusive events. We search for two mutually exclusive events, then look for a third one exclusive to the other two. Event  $A$  is a good start, since any event with a theft is exclusive to it, and only event  $D$  does not have a theft. Looking at  $B$ ,  $C$ , and  $E$ , we see that  $B$  has one theft in the second year and  $E$  doesn't, so they're mutually exclusive, and both are exclusive to  $A$ . **(A)**

**1.32.**  $P[A] = P[A \cap B] + P[A \cap C]$ , since the only way to have coverage  $A$  is in combination with exactly one of  $B$  and  $C$ . Similarly  $P[B] = P[B \cap A] + P[B \cap C]$  and  $P[C] = P[C \cap A] + P[C \cap B]$ . Adding up the three equalities, we get

$$P[A] + P[B] + P[C] = 2(P[A \cap B] + P[A \cap C] + P[B \cap C]) = \frac{1}{4} + \frac{1}{3} + \frac{5}{12} = 1$$

so the sum of the probabilities of two coverages is 0.5. But the only alternative to two coverages is no coverage, so the probability of no coverage is  $1 - 0.5 = 0.5$ . **(C)**

**1.33.** Use the inclusion-exclusion principle with 4 events, letter  $n$  is in envelope  $n$ ,  $n = 1, 2, 3, 4$ . The probability of at least one event occurring is  $1/4$ . The probability of two events is  $(1/4)(1/3) = 1/12$ . The probability of three events is  $(\frac{1}{4})(\frac{1}{3})(\frac{1}{2}) = \frac{1}{24}$ . And that is also the probability of all four events occurring. There are 6 ways to select 2 envelopes and 4 ways to select 3 envelopes.

The probability of at least one event occurring is

$$4\left(\frac{1}{4}\right) - 6\left(\frac{1}{12}\right) + 4\left(\frac{1}{24}\right) - \frac{1}{24} = \frac{5}{8} \quad \text{(D)}$$

1.34. The probability of  $U \cup V$  cannot be more than 1, and

$$P[U \cap V] = P[U] + P[V] - P[U \cup V] = 0.8 + 0.4 - P[U \cup V]$$

Using the maximal value of  $P[U \cup V]$ , we get the minimal value of  $P[U \cap V] \geq 0.8 + 0.4 - 1 = \boxed{0.2}$ .