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Exam FM Study Manual

17th Edition

Harold Cherry, FSA, MAAA and Wafaa Shaban, ASA, PH.D.

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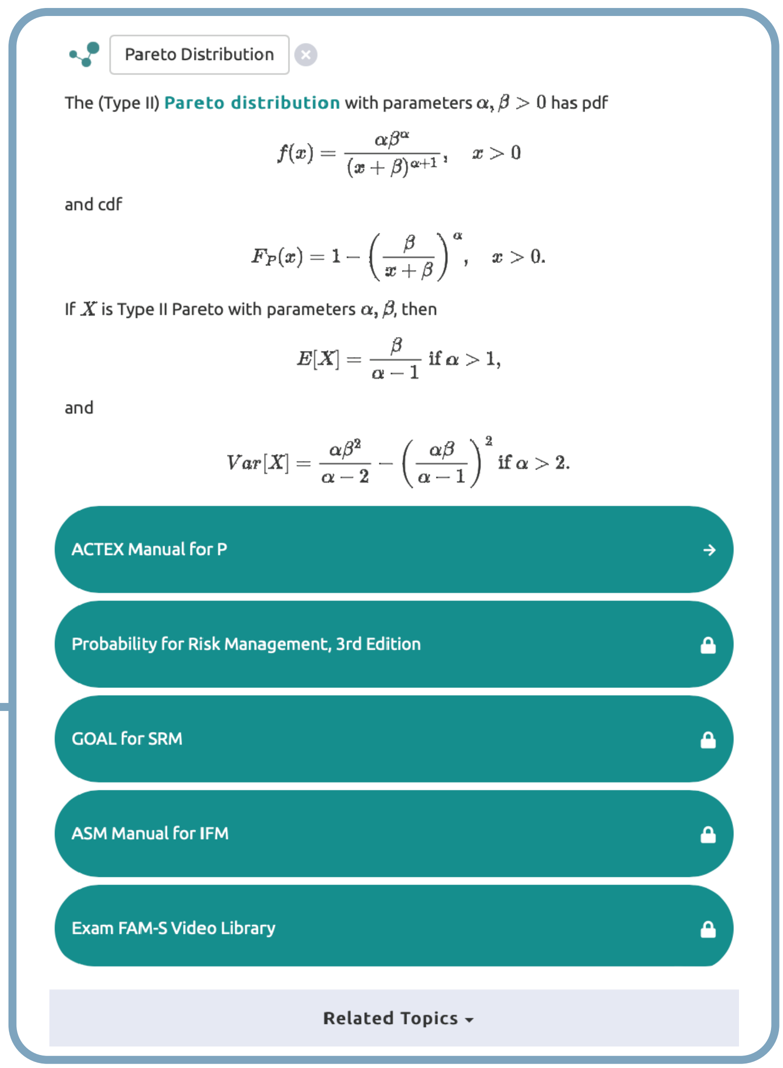

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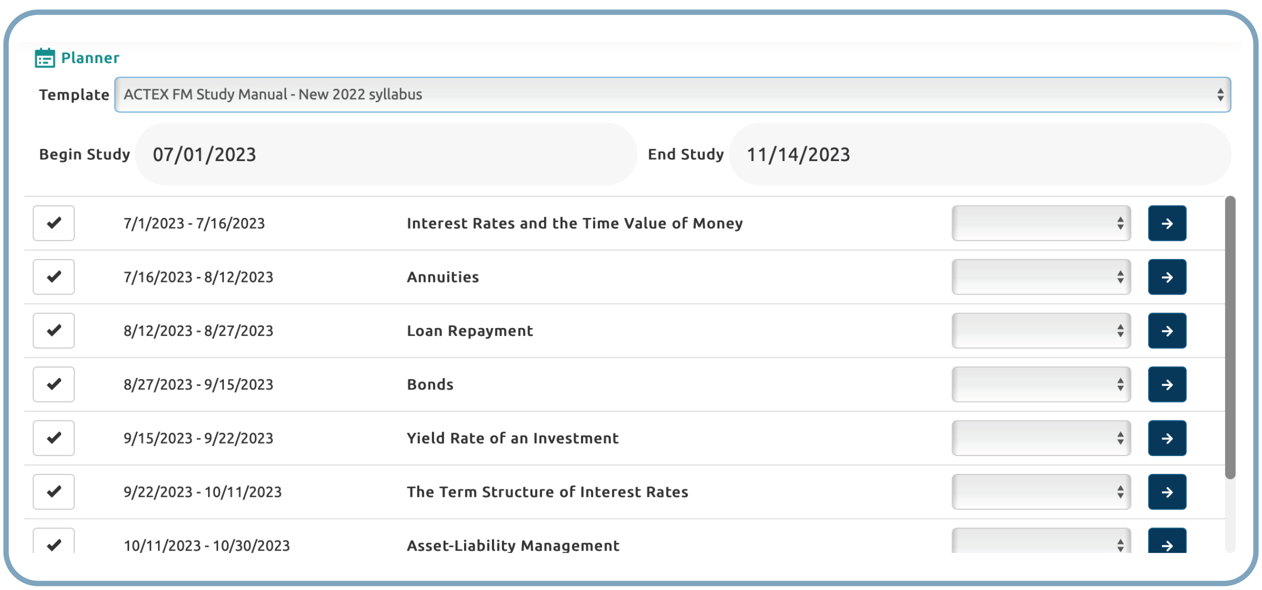

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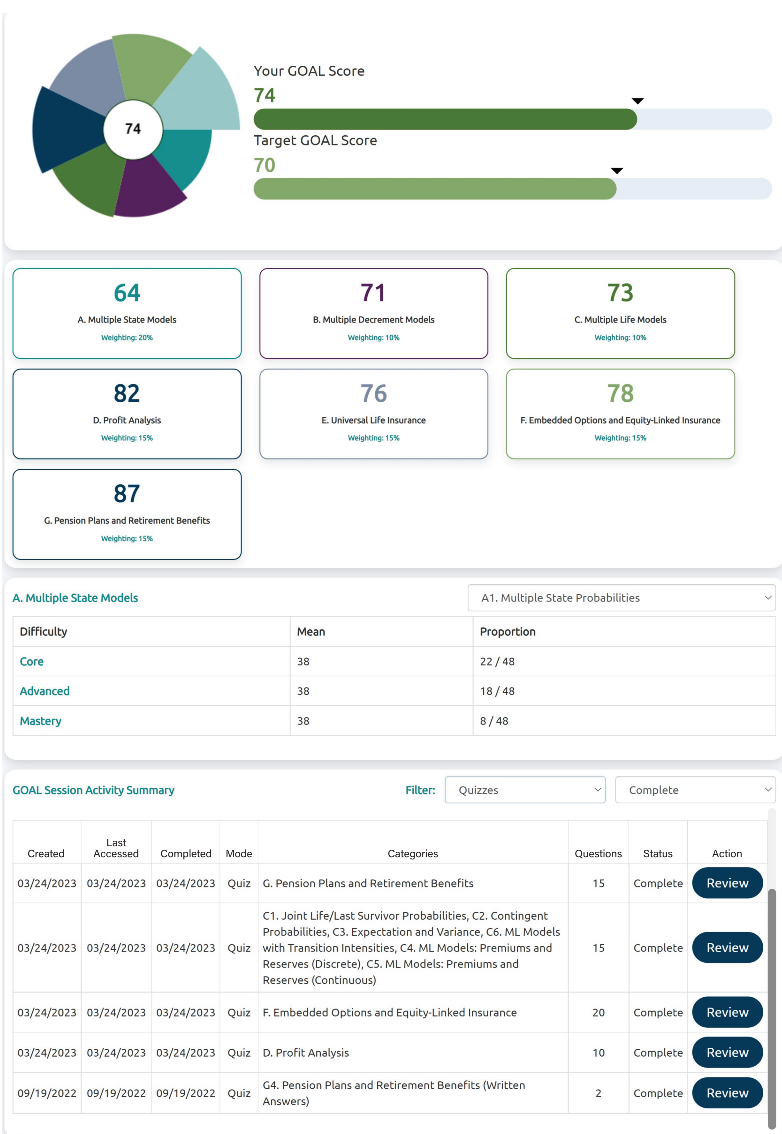

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Contents

Introduction

To the student: Please read this Introduction. It contains important information.

The 17^{th} edition of this manual has been revised in accordance with the new SOA syllabus beginning with the October 2022 exam. Also, this edition incorporates corrections of all known errata in the 16^{th} edition.

The 17^{th} edition consists of 9 chapters covering all of the material on the syllabus (Part I of the manual), followed by six original practice exams (Part II of the manual). Following each chapter or section in Part I, there are illustrative examples called "Stepping Stones" (see below). These examples are followed by a summary of the key concepts and formulas in the section, then by problems and solutions from actual past SOA/CAS examinations.

Goals of this Manual

- To explain the concepts of financial mathematics in a way that appeals to your intuition and common sense.
- To point out shortcuts and tricks that can get you to the answer more quickly.
- To warn you about common traps that students fall into and help you to avoid them.
- To provide you with hundreds of problems from past exams, with solutions.
- To provide you with original practice exams that will help prepare you for the real thing.

To highlight the concepts, tricks, shortcuts, and traps, you will see special symbols such as the following throughout the manual:

Concept Alert! **Shortcut Alert!**

Trick Alert! **Trap Alert!**

Problems

There is an old cliché in the real estate industry: What are the three most important factors in evaluating property? Answer: Location! Location! Location! If we had to say what the three most important factors are in passing this exam, they would be Problems! Problems! Problems! You must do a great variety of problems, preferably under time pressure, and especially as you get closer to the date of the examination.

Many of the problems in this manual are taken from past SOA and CAS examinations. We want to thank the two actuarial societies for their kind permission to publish these questions.

There are a number of points about past exam questions that you should be aware of:

- These questions, which date from the early 1980's, were created by different exam committees, under different syllabi, for exams of different length, etc. Thus, they can vary greatly in style, difficulty and emphasis of topics. In spite of this, you should find that solving prior exam problems, even very old ones, is helpful in preparing for the exam. It will expose you to a great variety of types of questions and approaches to solutions.
- This manual does not contain the questions and solutions from past FM exams published by the SOA. Currently, only the May 2005 and November 2005 exams have been published (but note that these two exams were very easy and followed an old syllabus that was in effect in 2005). You will also find a link to well over 200 sample questions and solutions, many of which are from past exams. ***You should definitely get these additional practice problems.*** You

can download them for free from the SOA website under Syllabus. We have not included them in this manual because we didn't want to waste space and expense by duplicating material that is freely available from another source.

- Since October 2022, the actual exam consists of 30 questions in CBT (computer based testing) format, with a $2\frac{1}{2}$-hour time limit.
- The order of the past exam questions that follow each topic in this manual is from the most recent to the oldest.
- There is a code following each question in this manual from a past exam that tells you which exam the question came from.
- Some of the questions have answer choices in ranges, such as "(A) Less than 11.3% (B) At least 11.3% but less than 11.5%," etc. These questions almost always come from CAS exams, which used this style for many years. Virtually all questions on the SOA/CAS Exam FM/2 since 2000 have had specific answer choices, rather than ranges.

To make it easier to locate them, the sections of the manual with past exam questions and solutions have running heads like this:

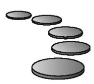
Past Exam Questions on Sections 1b to 1f

and

Solutions to Past Exam Questions on Sections 1b to 1f

"Stepping Stones"

Following each section of the manual, you will find illustrative examples, with solutions, that are called "Stepping Stones." They are indicated by this heading:



Stepping Stones

These examples are original to the manual and have a level of difficulty that runs from easy to moderate. They have two purposes:

- To make sure you understand the material that you have just read.
- To serve as "stepping stones" to more difficult exam-level problems.

Original Practice Exams

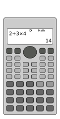
In addition to the questions from past exams, the manual contains six original full-length practice exams, with solutions. These exams consist of 30 questions each, with a $2\frac{1}{2}$-hour time limit, just like the actual exam. The distribution of questions by topic follows the SOA syllabus.

Take these exams in the order given. We believe that Practice Exam 6 is the most difficult. As we receive feedback from students about the difficulty of the six exams, we will post the information on the ASM website at www.studymanuals.com.

Calculators

Begin using your calculator immediately. Become thoroughly familiar with its operation. It should become like a trusted friend to you after awhile.

Our advice is to get a financial calculator for the exam, such as the BA II Plus© or BA II Plus Professional©. These calculators have special keys that are very useful for solving problem in financial mathematics. Which one should you use? You really don't need the Professional model, unless you are willing to spend extra money for some additional features that aren't essential for the exam. For this reason, we have based the Calculator Notes in this manual on the BA II Plus. You will find these notes at appropriate points in the manual, identified by this icon:



Please note that many students prefer to use one of the TI-30 MultiView series as their primary calculator because of the two-line display and superior algebraic capabilities. These students feel that they can even calculate certain interest functions, such as $a_{\overline{n}|}$, more quickly on the TI-30 series, even though the BA series has special keys for this function. They use the BA series as a backup calculator, mainly to calculate an unknown interest rate when the amounts of payments and their present value are given.

For a quick reference, here is the list of the Calculator Notes along with their corresponding page numbers:

Summary of Concepts

Following most sections in Part I, there is a summary of the key concepts and formulas covered in the section. To make it easier to locate them and for a quick review before taking your exam, the Summary of Concepts along with their corresponding page numbers are listed in the table below.

Studying for the Exam

Everyone is different. Some people like to study in the morning, some late at night. Some people can only study for a couple of hours before their brain begins to fry; others can study for long stretches. Some people can cram very effectively, but others need to cover the material over a period of several months. Almost everyone finds ways to procrastinate from time-to-time. You know your own study habits best, so build on your strengths and be aware of your weaknesses.

We recommend that you set up a study schedule right now. Determine how many hours you will realistically be able to put in between now and the exam. Many students use this rule-of-thumb: devote 100 hours of study time for each hour of exam time. For a $2\frac{1}{2}$-hour exam like FM/2, this would mean 250 hours. Of course, people vary considerably in how quickly they learn new material, so use this rule-of-thumb only as a rough guide.

When you set up your study schedule, be very aware of the SOA's allocation of questions by topic. The following table shows how many questions, on the average, you should expect for each topic, assuming that the SOA sticks to its own allocation when it constructs your particular computer-based exam. The table also shows which sections of the manual cover each topic.

Learning Objective	**Allocation by SOA**	**Midpoint of Allocation**	**Number of Questions (Based on a 30-Question Exam)**	**Chapters or Sections of the Manual that Cover the Topic**
1. Time Value of Money (Interest rate, discount rate, nominal rate, effective rate, force of interest, equation of value, inflation and real rate of interest, etc.)	5–15%	10%	3	Chapters 1, 2, 5
2. Annuities (annuity-immediate and due, perpetuity, m-thly, continuously, level payments, arithmetic and geometric increasing/decreasing, etc.)	20–30%	25%	7.5	Chapters 3, 4
3. Loans (Principal, interest, outstanding balance, final payment, amortization, refinancing, etc.)	15–25%	20%	6	Chapter 6
4. Bonds (Price, book value, amortization of premium, accumulation of discount, yield rate, coupon, face or par value, redemption value, etc.)	15–25%	20%	6	Chapter 7
5. General Cash Flows, Portfolios and Immunization (Yield rate, Macaulay and modified duration and convexity, spot rates, forward rates, yield curve, cash flow matching, Redington and full immunization, change in PV due to change in interest rate, etc.)	20–30%	25%	7.5	Chapters 5, 8, 9
Grand Total		100.0%	30	

The above table is based on the final syllabus for the October 2022 exam as posted by the SOA. If you are taking the exam in October 2022 or later, check the syllabus for the particular month and year of your exam, since the exam committee does make changes from time to time.

As you follow your schedule, will you fall behind from time-to-time? Of course. But if you have a schedule, at least you will know how far behind you are. This should spur you on to catch up.

Should you do all of the problems in a section before you move on to the next section? Our advice is to do as many problems as you can, but to move on if you are falling behind schedule; hopefully, you will catch up later.

You should leave room at the end of your schedule for at least three or four weeks of doing nothing but solving problems (and maybe a little bit of review of the topics you are having difficulty with). Be strict with yourself and work "by the clock." It's not important that you get the correct answer to a problem the first time that you do it. What *is* important is that you try the problem again a couple of days later and get it right the second (or even the third) time. (Mark off the problems that you don't get right and do batches of these problems a few days later.) This shows that you have really mastered the points being tested.

"Hitting a Brick Wall": Chapters 3 and 4

Some students find that when they attempt to cover the topics in Chapters 3 and 4, it's like "hitting a brick wall." There is a lot of material in these two chapters and it can be daunting if you are encountering it for the first time. If this happens to you, we have a couple of suggestions:

- Don't knock yourself out if you don't understand something. Move on to the next section and come back later on. You will find that as you spend more time on the material, you will mature in your understanding. Concepts that seemed difficult at first will seem easier when you revisit them.
- Don't try to do all of the problems at the end of each section the first time around. Do the first few, since they are the most recent, or do only the odd or even problems. Come back later and do the rest of the problems (or most of them). They will seem easier the second time around.

It's true that there are a lot of problems in these two chapters, about 200 in all, but this shows how important these topics have been on past exams.

Taking the Exam

The single most important rule about taking the exam is to *keep moving.* Don't get bogged down on any one question. Try to look at every question at least once. Be aware of your progress throughout the exam. Try to spend your time on the problems that are easier for you and drop the ones that are giving you too much trouble.

Since this exam has 30 questions and a $2\frac{1}{2}$-hour time limit, you have an average of about 5 minutes to spend on each question. You may be tempted to continue spending time on a problem because you have already invested, say, 5 or 6 minutes in it. We strongly suggest that you develop the discipline to drop the question at this point and move on. There may be an easier question waiting for you later on—and getting it right counts just as much as getting a hard question right. And you may have time to go back to the question later on and knock it off.

Your Comments

We welcome your comments, criticism, suggestions, and reports of any errata that you may find. Please e-mail us at *mail@studymanuals.com*

We wish you good studying and good luck on the exam. May the force of interest be with you.

Harold Cherry and Wafaa Shaban

Part I

Financial Mathematics

Chapter 3

Annuities

3a The Geometric Series Trap

Trap Alert!

In this chapter, we begin our study of annuities. An ***annuity*** is simply a regular series of payments. For example, let's determine the present value of an annuity of $1,000 payable at the end of each year for 10 years at 5% effective

$$PV = 1000(v + v^2 + \ldots + v^{10})$$
$$\text{where } v = 1.05^{-1}$$

You can see that the PV is the sum of a geometric progression. (Years ago, a college math prof of one of the authors disdainfully dismissed the subject of compound interest as "glorified geometric progressions." Of course, he never sat for the actuarial exams.)

Because ***geometric series*** do play a very important role in this subject, you ***must*** know how to sum them correctly. Maybe you think you do but our experience has been that many students fall into traps in applying the summation formula. When we ask students to sum series such as those in the following examples, we usually get four or five different answers, most of which are wrong. Try these examples yourself, but don't look at the solutions until you are finished.

Stepping Stones

Example 3.1

Sum the following series:

(a) $v^{10} + v^{15} + \ldots + v^{65} = ?$

(b) $(1+i)^{44} + (1+i)^{40}(1+r)^2 + (1+i)^{36}(1+r)^4 + \ldots + (1+i)^4(1+r)^{20} = ?$

Solution

The answers are $\boxed{\frac{v^{10}(1-v^{60})}{1-v^5}}$ for (a), and $\boxed{(1+i)^{44}\frac{[1-(1+i)^{-44}(1+r)^{22}]}{1-(1+i)^{-4}(1+r)^2}}$ for (b)

If you got different answers, first check whether they are algebraically equivalent to the above answers. Also, in answering (b) you could have used v^4 instead of $(1+i)^{-4}$.

You probably used the following formula for the sum of a geometric progression:

$$S = \frac{a(1-r^n)}{1-r}$$

where a is the first term, r is the common ratio and n is the number of terms. (In other words, this formula gives the sum of the series $a + ar + ar^2 + \ldots + ar^{n-1}$.) Some students remember this formula as $\frac{a-ar^n}{1-r}$, where the numerator can be interpreted as the first term minus the (fictitious) $(n+1)^{\text{st}}$ term, i.e., the term after the n^{th} or last term, ar^{n-1}.

A trap for some students is that for "n" they use the exponent of the last term in the series, or they make a similar substitution error. The way to avoid such traps is to remember the formula ***in words***, not symbols. You almost can't go wrong if you know the formula in the following way:

$$S = (\text{first term})\ \frac{[1-(\text{ratio})^{\text{no. of terms}}]}{1-\text{ratio}}$$

In series (a) above, the first term is v^{10}, the common ratio is v^5 and the number of terms is 12. (One way to count the number of terms—other than on your fingers—is $\frac{70-10}{5}=12$ where 70 is the exponent of the term ***after*** the actual last term, 10 is the exponent of the first term and 5 is the increment in the exponent from term to term.) If we think of the formula in terms of the ***words***, we have:

$$S = v^{10}\frac{[1-(v^5)^{12}]}{1-v^5} = v^{10}\frac{(1-v^{60})}{1-v^5}$$

For series (b), the first term is $(1+i)^{44}$, the ratio is $(1+i)^{-4}(1+r)^2$ (or $v^4(1+r)^2$) and the number of terms is 11 (which can be counted by using the exponents of the $(1+i)$ terms as $\frac{0-44}{-4}=11$ or by using the exponents of the $(1+r)$ terms as $\frac{22-0}{2}=11$.)

Again, substituting in the formula in terms of the *words,* we have:

$$\begin{aligned} S &= (1+i)^{44}\frac{\{1-[(1+i)^{-4}(1+r)^2]^{11}\}}{1-(1+i)^{-4}(1+r)^2} \\ &= (1+i)^{44}\frac{[1-(1+i)^{-44}(1+r)^{22}]}{1-(1+i)^{-4}(1+r)^2} \end{aligned}$$

which can be further simplifed to:

$$S = \frac{(1+i)^{44}-(1+r)^{22}}{1-(1+i)^{-4}(1+r)^2}$$

The geometric series formula is so simple and yet so treacherous. We urge you to remember it in the plain English version, using words instead of symbols. Practice saying it like a mantra (remember how you used to chant the quadratic formula?) and you will avoid falling into the traps that wait to ensnare you.

Example 3.2

Sum the following series: $v+1.03v^2+1.03^2v^3+1.03^3v^4+\cdots+1.03^{14}v^{15}$, where v is based on $i=5.25\%$. (Get a numerical result.)

Solution

Using the formula for the sum of a geometric progression in words, we have:

$$\text{Sum} = \frac{v[1-(1.03v)^{15}]}{1-1.03v}$$

Substituting $v=1/1.0525$:

$$\text{Sum} = \frac{1-\left(\frac{1.03}{1.0525}\right)^{15}}{1.0525\left(1-\frac{1.03}{1.0525}\right)} = \frac{1-\left(\frac{1.03}{1.0525}\right)^{15}}{.0225} = \boxed{\mathbf{12.30}}$$

Note: In Section 4j, we will see that this sum is the PV at 5.25% of a 15-year annuity-immediate with a first payment of 1 and with subsequent payments increasing geometrically at 3% (which could be referred to as an annuity with "inflation protection").

Example 3.3

Find the sum of the series in Example 3.2 if there are an infinite number of terms, rather than only 15 terms. (Get a numerical result.)

Solution

The sum of an infinite geometric progression is:

$$\text{Sum} = \frac{a}{1-r}$$

which has a limit if $-1<r<1$ of:

$$\text{Sum} = \frac{v}{1-1.03v} = \frac{1}{1.0525\left(1-\frac{1.03}{1.0525}\right)} = \frac{1}{.0225} = \boxed{\mathbf{44.4\bar{4}}}$$

Example 3.4

Find the sum of the series: $v + 2v^2 + 3v^3 + 4v^4 + \cdots + 20v^{20}$.

Solution

This is not a geometric series. (Sorry. This was a trick question.) In Section 4h, we will learn how to sum such a series. It represents the PV of an increasing annuity-immediate with payments in arithmetic progression.

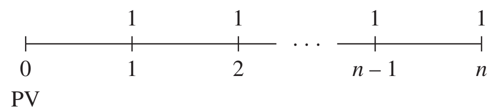
3b Annuity-Immediate and Annuity-Due

Almost everyone has had experience with annuities. If you pay rent, insurance premiums, mortgage payments, payments on a car or college loan, etc., the payments are an annuity. If you receive a regular paycheck, you are receiving an annuity. Someday you may receive Social Security benefits or a company pension for life (although traditional pension plans of this kind are on the wane). These, too, are types of annuities.

The stem of the word "annuity" is the Latin annu(us), for "yearly," since the earliest annuities had annual payments. So it's something of an oxymoron to speak of a "monthly annuity," a "quarterly annuity," etc., but that's how the word evolved.

Present Value of an Annuity-Immediate

Let's consider a very simple annuity - a ***payment of \$1 at the end of each year for n years***:



The PV of this annuity is:

$$PV = v + v^2 + \ldots + v^{n-1} + v^n$$

Assuming that you have mastered the summation of geometric series in the last section you should get:

$$PV = \frac{v(1 - v^n)}{1 - v}$$

You may recall from Section 1a that $1 - v = d = iv$, so if we substitute in the denominator, we get:

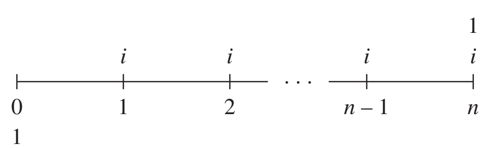
$$\boldsymbol{PV = \frac{1 - v^n}{i}}$$

An annuity having the first payment a year from now is called an "***annuity-immediate***." (It's a little strange to call it "immediate" when the first payment is not immediate.) The PV has a special symbol: $a_{\overline{n}|i}$. This is read "a angle n at i." (If the rate of interest is understood, we can just write $a_{\overline{n}|}$.)

At this point, you should practice getting numerical results. Turn to Calculator Notes #5 following Section 3c and do problems 22 to 26. If you are not using the BA II Plus, use your own calculator and compare your answers.

Here is a simple way to derive the formula for $a_{\overline{n}|}$ without using a summation formula: You deposit \$1.00 in a bank that credits interest at i effective. At the end of a year, you withdraw i, leaving \$1.00 in the account. You do the same thing at the end of years $2, 3 \ldots, n$, withdrawing i each year. Just after the n^{th} withdrawal, you still have \$1.00 in the account; you withdraw it and close the account.

This transaction can be diagrammed as follows:



If we let X represent the present value of an annuity-immediate of \$1.00 per annum for n years, the present value of the n withdrawals of i equals iX. Thus, we have the following equation of value as of time 0:

$$1 = iX + v^n$$

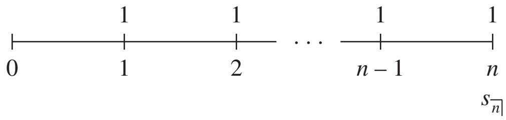
or

$$X = \frac{1 - v^n}{i}$$

Of course, the way we defined X, it is actually $a_{\overline{n}|}$, and we have derived its formula without making direct use of the summation formula.

Accumulated Value of an Annuity-Immediate

Let's determine the AV of an annuity-immediate of 1 for n periods. (Note that the AV of an annuity-immediate means the AV as of the date of the last payment or deposit.) The AV has the special symbol $s_{\overline{n}|}$.



Accumulating the payments to time n, starting with the last payment, we have:

$$s_{\overline{n}|} = 1 + (1+i) + \ldots + (1+i)^{n-2} + (1+i)^{n-1}$$

(Note carefully that this geometric series starts with 1 and ends with $(1+i)^{n-1}$.)

Summing the series, we have:

$$s_{\overline{n}|} = \frac{1[1-(1+i)^n]}{1-(1+i)} = \frac{(1+i)^n - 1}{i}$$

Of course, rather than summing a geometric series again, we could have relied on the relationship between $a_{\overline{n}|}$ and $s_{\overline{n}|}$. After all, these symbols represent the value of the ***same*** payments on ***two different dates*** which are n years apart. Thus, we have:

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|} = (1+i)^n \left(\frac{1-v^n}{i}\right) = \frac{(1+i)^n - 1}{i}$$

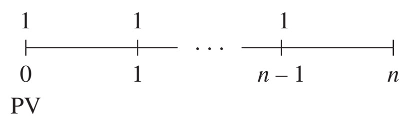
since $(1+i)^n$ and v^n are reciprocals.

(At this point, we caution you that $s_{\overline{n}|}$ is a deceptively simple interest symbol that can easily get you into trouble. More on this in the next section.)

You should now turn to Calculator Notes #5 again and do problems 27 to 29.

Present Value of an Annuity-Due

Suppose we want the PV of an annuity with n payments where the first payment is today, rather than a year from now



$$PV = 1 + v + \ldots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

When the first payment is today, an annuity is called an ***annuity-due***. The PV has the symbol $\ddot{a}_{\overline{n}|}$, which is read "a double-dot angle n."

Mnemonic: Note that the formula for an annuity-***i***mmediate has an ***i*** in the denominator:

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

while the formula for an annuity-***d***ue has a ***d*** in the denominator:

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

i for ***i***mmediate and ***d*** for ***d***ue.

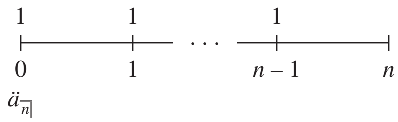
Relationships Between $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$

There are obvious relationships between the "immediate" and "due" forms of an annuity. The symbols $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ both represent the value of the same payments, but $\ddot{a}_{\overline{n}|}$ is equal to the value of $a_{\overline{n}|}$ a year later. Thus, we have:

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$$

(If you have any doubt about this, ask yourself which you would rather have today: a series of payments that begins today or the ***same*** series of payments beginning a year from now. Clearly, you want the payments to begin as soon as possible, so $\ddot{a}_{\overline{n}|}$ is greater than $a_{\overline{n}|}$; in particular, $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.)

There is another simple relationship between the "immediate" and "due" forms. To derive it, we will play a game that we call "now you see it, now you don't." (We'll play it some more in Section 3d.) Here is the diagram for the annuity due again:



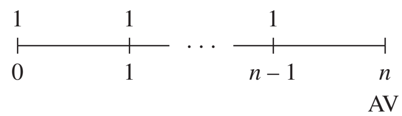
Now, cover up the payment at time 0 (mentally or with your finger). What's left? Answer: An annuity-immediate with $(n-1)$ payments. So we can write the PV of an annuity-due as the payment of \$1.00 at time 0 plus the PV of an annuity-immediate with one less payment:

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

Now turn to Calculator Notes #5 and do problems 30 and 31 for practice in getting numerical results for problems solving $\ddot{a}_{\overline{n}|}$.

Accumulated Value of an Annuity-Due

By the AV of an annuity-due, we mean the AV one period ***after*** the last payment, as in the following diagram:



The symbol for this AV is $\ddot{s}_{\overline{n}|}$. Clearly, the series is:

$$\ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + ... + (1+i)^{n-1} + (1+i)^n$$

(Note very carefully that the series begins with $(1+i)$ and ends with $(1+i)^n$, while the series for $s_{\overline{n}|}$ begins with 1 and ends with $(1+i)^{n-1}$.)

Summing the series, we have:

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)[1-(1+i)^n]}{1-(1+i)}$$

If we reverse signs of numerator and denominator and bring $(1+i)$ "downstairs" in the denominator as v, we have:

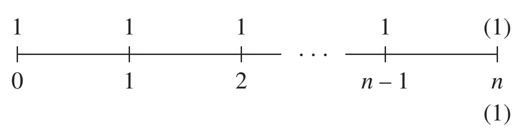
$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{iv} = \frac{(1+i)^n - 1}{d}$$

Earlier, we derived certain relationships between the "immediate" and "due" forms for present value, and we can do the same for accumulated value.

Which is worth more, $\ddot{s}_{\overline{n}|}$ or $s_{\overline{n}|}$? Clearly, you would rather have the AV of a series of deposits ***one year after*** the last deposit than the AV of the same deposits on ***the date of the last deposit***. In particular, $\ddot{s}_{\overline{n}|}$ is equal to the value of $s_{\overline{n}|}$ one year later, i.e.,

$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}$$

Also, we could play "now you see it, now you don't" (or perhaps more accurately, "now you don't see it, now you do"):



What we have done is to place a fictitious deposit at time n above the time line but to immediately withdraw it at time n below the time line. This leaves the AV of the original annuity exactly the same. Now look at all of the deposits, including the fictitious one. Their AV as of time n is the AV of an annuity-immediate with $(n+1)$ payments, i.e., $s_{\overline{n+1}|}$. But this AV overstates the true AV by 1, which we must deduct. Thus, we have the following relationship:

$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1.$$

Now turn to Calculator Notes #5 and do problems 32 and 33 for practice in getting numerical results for problems involving $\ddot{s}_{\overline{n}|}$.

Problems Involving Both Present Values and Accumulated Values

Turn to Calculator Notes #5 and do problems 34 to 36 as examples of problems where both the PV and AV of annuities are involved.

Stepping Stones

Example 3.5

Given that $a_{\overline{n}|} = 7.912718$ and $\ddot{a}_{\overline{n}|} = 8.268790$, determine n.

Solution

Since $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$, $1+i = \ddot{a}_{\overline{n}|}/a_{\overline{n}|} = 8.268790/7.912718 = 1.045$ and $i = 4.5\%$. We can determine n by using the calculator:

4.5 [I/Y] 7.912718 [PV] 1 [+/–] [PMT] [CPT] [N]

$n =$ **10**

Example 3.6

Given $a_{\overline{n}|} = 8.50$ and $s_{\overline{n}|} = 16.63$, determine i.

Solution

Since $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$, $(1+i)^n = s_{\overline{n}|}/a_{\overline{n}|} = 16.63/8.50 = 1.956471$.

Now, $s_{\overline{n}|} = [(1+i)^n - 1]/i$. Since we have numerical values for $s_{\overline{n}|}$ and $(1+i)^n$, we can determine i:

$$16.63 = (1.956471 - 1)/i$$

$$i = \boxed{5.75\%} \quad \text{(to 2 decimals)}$$

3c The Great Confusion: Annuity-Immediate and Annuity-Due

Students sometimes get confused by the terms “annuity-immediate” and “annuity-due,” and the symbols for the present value and accumulated value of these annuities. In this section, we will consider a few examples that should help clear up this confusion.

Stepping Stones

Example 3.7

Suppose you are told that an annuity pays \$1.00 on January 1 of each year from 2020 through 2029. Let’s draw a time diagram for these 10 payments:

Here’s the question: Is this an annuity-immediate or an annuity-due?

Your first impulse would probably be to say that this is an annuity-due. After all, payments are made at the beginning of each year. Isn’t that the definition of an annuity-due?

But what if you were asked to determine the present value of this annuity on 1/1/19, one year before the first payment? Wouldn’t you say that you were being asked to find the PV of an annuity-immediate, $a_{\overline{10}|}$?

And what if you were asked to determine the present value of the same annuity on 1/1/20, the date of the first payment? Wouldn’t you say that you were being asked to find the PV of an annuity-due, $\ddot{a}_{\overline{10}|}$?

It seems as if the same annuity can be an annuity-immediate or an annuity-due, *depending on the valuation date.*

That’s exactly right.

Unfortunately, many textbooks define an annuity-immediate and an annuity-due based on whether the payments are made at the "end of each period" or "the beginning of each period." A typical definition might be:

> If payments are made at the end of each period, the annuity is called an "annuity-immediate." If payments are made at the beginning of each period, the annuity is called an "annuity-due."

This definition is deceptive. A more precise definition is:

> An annuity is called an "annuity-immediate" if, in determining its present value, the valuation date is *one period before* the first payment (symbol $a_{\overline{n}|}$). An annuity is called an "annuity-due" if, in determining its present value, the valuation date is *on* the date of the first payment (symbol $\ddot{a}_{\overline{n}|}$).

It may seem as if we are making a big fuss over something that is pretty obvious. But you would be surprised at how often this can lead to confusion, especially when we need to find the *accumulated value* of an annuity.

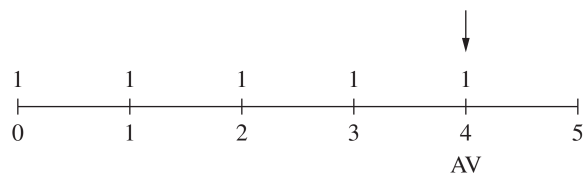
Let's take an example of an accumulated value.

Example 3.8

$1.00 is deposited at the beginning of each year for 5 years at i effective. How much is in the account on the date of the last deposit? (Use a symbol.)

Think for a moment and write your answer down.

Hopefully, you wrote $s_{\overline{5}|}$, not $\ddot{s}_{\overline{5}|}$. Remember, you were asked for the AV of this annuity *on the date of the last deposit.* If you drew a time diagram, it might look like this:



Note that we have placed "AV" at *time 4,* which is the date of the last deposit.

If we write out this AV term by term, starting with the deposit that was just made at time 4, we get:

$$AV = 1 + (1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4$$

Isn't this the definition of $s_{\overline{n}|}$, with $n = 5$, that we just covered in Section 3b?

Example 3.9

$1.00 is deposited at the beginning of each year for 5 years at i effective. How much is in the account one year after the last deposit? (Use a symbol.)

Clearly, the answer is $\ddot{s}_{\overline{n}|}$. We could use the same diagram as in the previous example, except that we would place "AV" at time 5, one year after the last deposit. Writing out this AV term by term:

$$AV = (1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4 + (1+i)^5$$

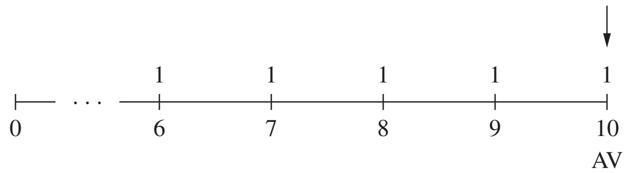
This is the definition of $\ddot{s}_{\overline{n}|}$, with $n = 5$, that we covered in Section 3b.

When it comes to accumulated values, the definition of an annuity-immediate or an annuity-due must take into account the timing of the payments *with respect to the valuation date,* just as in the case of present values. We suggest the following definition:

> An annuity is called an "***annuity-immediate***" if, in determining its accumulated value, the valuation date is *on* the date of the last payment (symbol $s_{\overline{n}|}$). An annuity is called an "***annuity-due***" if, in determining its accumulated value, the valuation date is *one period after* the date of the last payment (symbol $\ddot{s}_{\overline{n}|}$).

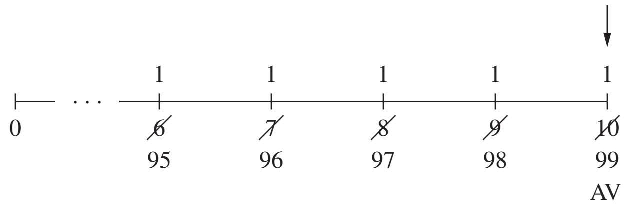
Example 3.10

$1.00 is deposited at the end of the 6th through 10th years. How much is in the account on the date of the last deposit?

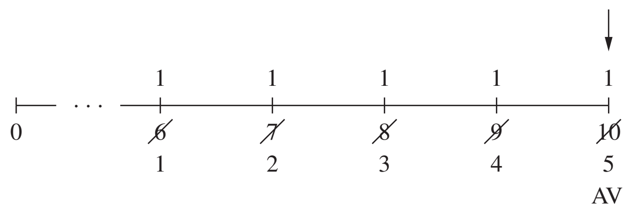


The answer is $s_{\overline{5}|}$. It's certainly not $s_{\overline{10}|}$ or $(s_{\overline{10}|} - s_{\overline{5}|})$.

The time scale can cause confusion. Would it change the answer to this question if we renumbered the dates as follows?



We think you would agree that the answer is not $s_{\overline{99}|}$ or $(s_{\overline{99}|} - s_{\overline{94}|})$. In fact, whatever dates we use, as long as the diagram shows 5 annual payments, with the AV on the date of the last payment, the answer is always $s_{\overline{5}|}$. This includes changing the dates as follows:



In this version of the time diagram, it's clear that the AV is equal to $s_{\overline{5}|}$, just as for the two preceding diagrams. On a term by term basis, we have the following for all three diagrams:

$$AV = 1 + (1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4 = s_{\overline{5}|}$$

The accumulated value of 5 annual deposits of $1.00 as of the date of the last deposit is always $s_{\overline{5}|}$, whether the payments are labeled as being at time 0 to 4, 1 to 5, 6 to 10, 95 to 99, etc.

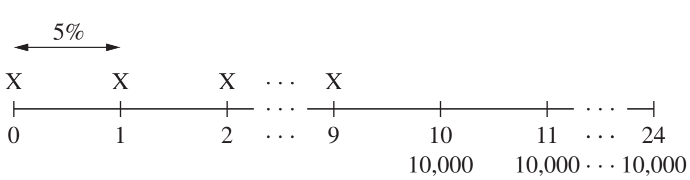
Example 3.11

Joe makes 10 annual deposits of X each into a fund earning 5% effective. These deposits accumulate to an amount that is just sufficient to allow him to withdraw $10,000 annually for 15 years, with the first withdrawal one year after the last deposit. Set up a time diagram, choose a comparison date, and write an equation of value. Then get a numerical value for X.

Solution

You are not told whether the deposits or withdrawals are made at the beginning or end of each year. However, because you know that the first withdrawal is one year after the last deposit, you have enough information to determine X.

It's up to you as to how you set up the time diagram. Many students would say, "Well, I might as well assume that the deposits start right away, so I'll place X at time $0, 1, 2, \ldots, 9$." If you take this approach, your time diagram would look like this:



Note that the first withdrawal has been placed one year after the last deposit.

What comparison date would you choose? Many students would think that since the deposits have been placed at the beginning of each year, they should use time 10 as the comparison date. They would then write the following equation of value:

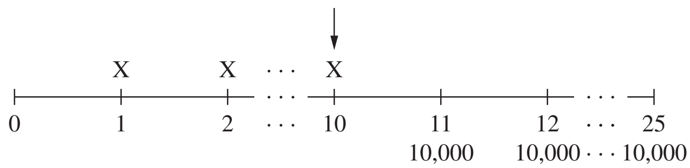
$$X\ddot{s}_{\overline{10}|} = 10{,}000\ddot{a}_{\overline{15}|}$$

There's nothing wrong with this approach from a theoretical point of view. However, we try to avoid using the "double-dot" form of annuities if possible, for the practical reasons given in Calculator Notes #5. (See problems 30 to 33.) Since "double-dots cancel" (this is covered in a later section), we can simply remove them in the above equation. (What this really means is that we cancel d in the denominators of both sides of the equation and replace d by i.) This gives us the following equivalent equation of value:

$$Xs_{\overline{10}|} = 10{,}000a_{\overline{15}|}$$

Of course, we could have gotten directly to this equation by using *time 9* as the comparison date, wouldn't you agree?

An even better approach would be to set up the time diagram in the first place as follows:



Using time 10 as the comparison date, we get the above equation of value.

We think that this approach is the best way to avoid making any mistakes. You will learn how to set up your time diagrams in the most convenient way as you get more experience in doing problems.

Using the methods shown in Calculator Notes #5, you should get $X =$ **$8,252.30**.

Calculator Notes #5: Annuities

When you do problems involving annuities, it's easier to use the ***TVM keys*** than it is to use first principles. You will be using all of the TVM keys, including the [PMT] key. This is the register for the level annuity payment.

If you enter 3 of the 4 variables [N], [I/Y], [PMT] and [PV], you can calculate the unknown 4th variable. You can do the same thing if [FV] is one of the variables, rather than [PV].

Problems 22 to 26 involve $a_{\overline{n}|}$

22. Find the PV of a 10-year annuity-immediate with annual payments of $1,200 at an effective interest rate of 6% per annum.

    The keystrokes are:

    [2nd] [CLR TVM]

    (Just a reminder to always clear the TVM registers at the start of a new problem.)

    10 [N] 6 [I/Y] 1,200 [PMT] [CPT] [PV]

    The answer is −$8,832.10 to 2 decimals.

23. Find the PV of an annuity-immediate of $100 per annum for 15 years at 4% effective.

    15 [N] 4 [I/Y] 100 [PMT] [CPT] [PV]

    The answer is −$1,111.84.

24. Find the PV of an annuity-immediate with payments of $50 every 6 months for 10 years at a nominal rate of interest of 4% compounded semiannually.

    20 [N] 2 [I/Y] 50 [PMT] [CPT] [PV]

    The answer is −$817.57.

25. A 12-year annuity-immediate is purchased for $100,000 at 5.5% effective. What is the level annual payment provided by this annuity?

    12 [N] 5.5 [I/Y] 100,000 [+/−] [PV][CPT] [PMT]

    The answer is $11,602.92.

26. A loan of $1,000 is to be repaid by equal quarterly installments of X at the end of each quarter over a 3-year period at a nominal rate of interest of 4% compounded quarterly. Determine X.

    12 [N] 1 [I/Y] 1,000 [PV][CPT] [PMT]

    The answer is −$88.85.

Problems 27 to 29 involve $s_{\overline{n}|}$

27. John deposits $250 at the end of each month for 5 years in an account that credits interests at a nominal rate of 6% per annum compounded monthly. How much is in his account on the date of the last deposit?

    In this problem, one of the four variables is [FV], rather than [PV]. Since we want the accumulated value on the date of the last deposit, the answer in symbolic form is $250s_{\overline{60}|}$ at 0.5%. The keystrokes are:

    60 [N].5 [I/Y] 250 [+/−] [PMT] [CPT] [FV]

    The answer is $17,442.51.

28. $1,200 is deposited in an account on January 1 of each year from 2005 to 2015, inclusive, at 3% effective. How much is in the account on January 1, 2015?

    In symbolic form, the answer is $1,200s_{\overline{11}|.03}$. (Note that there are 11 deposits from 2005 to 2015, inclusive, and that the question asks for the AV on the date of the last deposit (January 1, 2015). Thus, this is the accumulated value of an annuity-immediate.) The keystrokes are:

    11 [N] 3 [I/Y] 1,200 [+/−] [PMT] [CPT] [FV]

    The answer is $15,369.35.

29. What quarterly deposit for 6 years will accumulate to $22,000 on the date of the last deposit at a nominal rate of interest of 7% compounded quarterly?

    In symbolic form, the answer is the solution for X in $Xs_{\overline{24}|1.75\%} = 22{,}000$.

    24 [N] 1.75 [I/Y] 22,000 [FV] [CPT] [PMT]

    The answer is −$745.48.

Problems 30 and 31 involve $\ddot{a}_{\overline{n}|}$

30. Find the PV of an annuity-due with payments of $1,800 every 6 months for 8 years at a nominal rate of interest of 5% per annum compounded semiannually.

    This problem asks for the PV of an annuity with payments at the *beginning* of each 6-month period for 16 periods, at an interest rate of 2.5% per period. The answer in symbolic form is $1{,}800\ddot{a}_{\overline{16}|.025}$ or $1{,}800(1.025)a_{\overline{16}|}$ or $1{,}800(1 + a_{\overline{15}|})$.

    The calculator has a [BGN] key, secondary to the [PMT] key, which can be used to compute an annuity-due. We will do the problem with and without using the [BGN] key.

a. Without using the [BGN] key

(Make sure that you are in [END] mode. If you are in [BGN] mode, "BGN" will show in the upper right hand corner of the display. To change to [END] mode, see (b) below.)

We will calculate $1{,}800a_{\overline{16}|}$ at 2.5% and then multiply by 1.025. (This is based on the relationship $\ddot{a}_{\overline{16}|} = (1 + i)a_{\overline{16}|}$.)

16 [N] 2.5 [I/Y] 1,800 [PMT] [CPT] [PV][×] 1.025 [=]

The answer is −$24,086.48.

b. Using the [BGN] key

Trap Alert!

It is true that you can use the [BGN] key to easily calculate an annuity-due. But we have to warn you that it's very easy to forget that you are in [BGN] mode when you move on to another problem. *The calculator will stay in [BGN] mode even if you clear the TVM keys by pressing* [2nd] [CLR TVM], *and it will remain in this mode until you actively change it!*

How do you avoid this trap? Our own preference is not to use the [BGN] key if we can help it. In problem 30, you could compute an annuity-immediate and multiply by $(1 + i)$, as we did above. You could also compute the PV of an annuity-immediate with one less payment and then add a payment. (This is based on the relationship $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$.)

If you do decide to regularly use the [BGN] key to evaluate an annuity-due, practice a lot, *and make sure that when you move on to another problem, you are in the correct mode for that problem.*

To put the calculator in [BGN] mode, do the following:

[2nd] [BGN] [2nd] [SET]

If you repeat these 4 keystrokes, the left-hand side of the display will alternate between "END" and "BGN". When it shows "BGN", return to standard calculator mode by pressing [2nd] [QUIT]. You are now in [BGN] mode, and "BGN" will be shown in small font in the upper right-hand corner of the display.

The keystrokes for problem 30 are:

16 [N] 2.5 [I/Y] 1,800 [PMT] [CPT] [PV]

The answer is −$24,086.48, the same as before. Note that these are the same keystrokes as in (a), except that the last step of multiplying by 1.025 is omitted. Of course, this is because you are in [BGN] mode, which directly calculates an annuity-due.

31. Mary deposits \$15,000 today in a bank crediting interest at a nominal rate of 5% compounded monthly. This deposit is just sufficient to permit her to make monthly withdrawals of X for 6 years, first withdrawal today. Determine X.

X is the solution of $X\ddot{a}_{\overline{72}|5/12\%} = 15{,}000$. (Note that by convention, "monthly withdrawals for 6 years" means 72 withdrawals, not 73, even if the first withdrawal is today.)

a. Without using the [BGN] key

Make sure that you are in [END] mode. The keystrokes are:

72 [N] 5 [÷] 12 [=] [I/Y] 15,000 [PV][CPT] [PMT]

At this point, you have calculated the withdrawal that Mary could make at the *end* of each month, since you are in [END] mode. If Mary makes each withdrawal one month earlier, you must multiply this amount by v, where $v = (1+i)^{-1}$ and $i = 5/12\%$, the effective monthly rate. The keystrokes are:

.05 [÷] 12 [+] 1 [=] [1/x] [×] [RCL] [PMT] [=]

The answer is $-\$240.57$.

b. Using the [BGN] key

Make sure you are in [BGN] mode. Use the same keystrokes as in (a), except for the last step of multiplying by v. The answer is $-\$240.57$, as before.

Problems 32 and 33 involve $\ddot{s}_{\overline{n}|}$

32. Determine the AV of 13 annual deposits of \$1,429 one year after the last deposit, at 2.10% effective.

In symbolic form, the answer is $1{,}429\ddot{s}_{\overline{13}|2.10\%}$ or $1{,}429(1+i)s_{\overline{13}|}$ or $1{,}429(s_{\overline{14}|} - 1)$.

a. Without using the [BGN] key

Make sure that you are in [END] mode. The keystrokes are:

13 [N] 2.10 [I/Y] 1,429 [+/−] [PMT] [CPT] [FV] [×] 1.021 [=]

The answer is \$21,551.03.

b. Using the [BGN] key

Make sure that you are in [BGN] mode. Use the same keystrokes as in (a) but omit the last step of multiplying by 1.021. The answer is the same as before.

33. Andrea makes deposits of \$100 on the first day of each month in calendar years 2010 through 2015, inclusive, at a nominal rate of 7% per annum convertible monthly. How much is in her account on January 1, 2016?

This is the AV of an annuity-due, since the date of evaluation (January 1, 2016) is one month after the last deposit. There are 72 deposits. (There are 6 years from 2010 through 2015, inclusive.)

a. Without using the [BGN] key

You must be in [END] mode. The keystrokes are:

72 [N] 7 [÷] 12 [=] [I/Y] 100 [+/−] [PMT] [CPT] [FV]

At this point, you have calculated $100s_{\overline{72}|}$, so you must multiply by 1 plus the effective monthly rate:

.07 [÷] 12 [+] 1 [×] [RCL] [FV] [=]

The answer is \$8,968.10.

b. Using the [BGN] key

You must be in [BGN] mode. The keystrokes are the same as in (a), except that you omit the last step of multiplying by $(1+i)$.

Problems involving both the PV and AV of an annuity

34. Ed makes 15 annual deposits of X, which are just sufficient to allow him to make 8 annual withdrawals of $1,000, first withdrawal one year after last deposit. Interest is at 5% effective. Determine X.

    The most convenient comparison date is the date of the last deposit. X is the solution of $Xs_{\overline{15}|} = 1{,}000a_{\overline{8}|}$ at 5%. First, we calculate [PV], the right-hand side of the equation. Since this is the amount that the 15 deposits of X must accumulate to, we enter it in [FV] and proceed from there. The keystrokes are:

    8 [N] 5 [I/Y] 1,000 [PMT] [CPT] [PV] [2nd] [CLR TVM] [FV] 15 [N] 5 [I/Y] [CPT] [PMT]

    The answer is 299.52.

    Note the following:

    1. After you clear the TVM registers, the PV (= −6,463.21) is still in the display, so it can be entered in [FV] simply by pressing that key.
    2. An alternative is to omit the step [2nd] [CLR TVM]. You would then use the keystrokes:

       [FV] 15 [N] 0 [PV] [CPT] [PMT]

    Again, the answer is 299.52.

    Note that if you don't clear the TVM registers after calculating $1{,}000a_{\overline{8}|}$, you don't have to re-enter the 5% interest rate *but you have to zero-out* [PV]*, since any values from the previous calculation are still in the registers.* The safest thing to do is to clear the TVM registers, even if you have to re-enter one or more values that don't change. You be the judge.

35. Same as problem 34, except that the interest rate drops to 4% effective immediately after the last deposit.

    The keystrokes are exactly the same as in problem 34, except that the interest rate is entered as 4% the first time.

    The answer is 312.01. Note that because a lower interest rate (4%) is earned during the withdrawal period, the deposit X must be larger than it was in problem 34 (312.01 vs. 299.52).

36. Deposits of $100 are made every month for 5 years into an account crediting interest at a nominal rate of 9% convertible monthly. Starting one month after the last deposit, monthly withdrawals of X are made for 10 years, exhausting the account. Determine X.

    The most convenient comparison date is the date of the last deposit. The equation of value is $100s_{\overline{60}|} = Xa_{\overline{120}|}$ at .75%. First, we calculate the left-hand side of the equation as [FV]. This is entered in [PV] for the second part of the calculation. The keystrokes are:

    60 [N] .75 [I/Y] 100 [PMT] [CPT] [FV] [2nd] [CLR TVM] [PV] 120 [N] .75 [I/Y] [CPT] [PMT]

    The answer is 95.54.

Summary of Concepts and Formulas in Sections 3a to 3c

1. The best way to remember the formula for the ***sum of a geometric progression*** is in words:

$$\text{Sum of geometric progression} = (\text{first term})\frac{[1-(\text{ratio})^{\text{no. of terms}}]}{1-\text{ratio}}$$

2. (a) ***PV of an annuity-immediate***:

$$\begin{aligned} a_{\overline{n}|} &= v + v^2 + \cdots + v^n \\ &= \frac{1-v^n}{i} \end{aligned}$$

   (b) ***AV on the date of the last payment***:

$$\begin{aligned} s_{\overline{n}|} &= 1 + (1+i) + \cdots + (1+i)^{n-1} \\ &= \frac{(1+i)^n - 1}{i} \end{aligned}$$

3. (a) ***PV of an annuity-due***:

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= \frac{1-v^n}{d} \end{aligned}$$

   (b) ***AV one year after last payment***:

$$\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i) + (1+i)^2 + \cdots + (1+i)^n \\ &= \frac{(1+i)^n - 1}{d} \end{aligned}$$

4. Note the "i" in the denominator for "immediate" and the "d" for "due."

5. Relationships between annuity-immediate and annuity-due:

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= (1+i)a_{\overline{n}|} = 1 + a_{\overline{n-1}|} \\ \ddot{s}_{\overline{n}|} &= (1+i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1 \end{aligned}$$

6. Avoid the $s_{\overline{n}|}$ trap: The AV *on the date of the last payment* is $s_{\overline{n}|}$ (if there are n payments), regardless of how the time diagram is numbered. The AV *one year after the last payment* is $\ddot{s}_{\overline{n}|}$ (if there are n payments).

Past Exam Questions on Sections 3a to 3c

1. A man turns 40 today and wishes to provide supplemental retirement income of 3000 at the beginning of each month starting on his 65th birthday. Starting today, he makes monthly contributions of X to a fund for 25 years. The fund earns a nominal rate of 8% compounded monthly. Each 1000 will provide for 9.65 of income at the beginning of each month starting on his 65th birthday until the end of his life. Calculate X. [11/01 #27]

   (A) 324.73 (B) 326.89 (C) 328.12 (D) 355.45 (E) 450.65

2. Susan and Jeff each make deposits of 100 at the end of each year for 40 years.

   Starting at the end of the 41st year, Susan makes annual withdrawals of X for 15 years and Jeff makes annual withdrawals of Y for 15 years. Both funds have a balance of 0 after the last withdrawal.

   Susan's fund earns an annual effective interest rate of 8%. Jeff's fund earns an annual effective interest rate of 10%.

   Calculate $Y - X$. [SAMPLE/00 #27]

   (A) 2792 (B) 2824 (C) 2859 (D) 2893 (E) 2925

3. Jeff and John shared equally in an inheritance.

   Using his inheritance, John immediately bought a 10-year annuity-due with annual payments of 2500 each.

   Jeff put his inheritance in an investment fund earning an annual effective interest rate of 9%. Two years later, Jeff bought a 15-year annuity-immediate with annual payment of Z.

   The present value of both annuities was determined using an annual effective interest rate of 8%.

   Calculate Z. [SOA 11/96 #4]

   (A) 2330 (B) 2470 (C) 2515 (D) 2565 (E) 2715

4. At time $t = 0$, Paul deposits P into a fund crediting interest at an effective annual interest rate of 8%. At the end of each year in years 6 through 20, Paul withdraws an amount sufficient to purchase an annuity-due of 100 per month for 10 years at a nominal interest rate of 12% compounded monthly. Immediately after the withdrawal at the end of year 20, the fund value is zero.

   Calculate P. [SOA 11/93 #4]

   (A) 41,000 (B) 42,000 (C) 43,000 (D) 44,000 (E) 45,000

5. At an annual effective interest rate of 6.3%, an annuity immediate with $4N$ level annual payments of 1,000 has a present value of 14,113.

   Determine the fraction of the total present value represented by the first set of N payments and the third set of N payments combined. [SOA 5/93 #4]

   (A) 57% (B) 60% (C) 63% (D) 66% (E) 69%

6. An investment requires an initial payment of 10,000 and annual payments of 1,000 at the end of each of the first 10 years. Starting at the end of the eleventh year, the investment returns five equal annual payments of X.

   Determine X to yield an annual effective rate of 10% over the 15-year period. [SOA 11/92 #9]

   (A) 10,750 (B) 10,900 (C) 11,050 (D) 11,200 (E) 11,350

7. (Final balloon payments are explained in Section 6d.) You are given an annuity-immediate with 11 annual payments of 100 and a final balloon payment at the end of 12 years. At an annual effective interest rate of 3.5%, the present value at time 0 of all the payments is 1,000.

   Using an annual effective interest rate of 1%, calculate the present value at the beginning of the ninth year of all remaining payments. [SOA 11/89 #6]

   (A) 412 (B) 419 (C) 432 (D) 439 (E) 446

8. (Final payments are explained in Section 6d.)

Kimberley is told that she can receive a 250,000 death benefit from her husband's life insurance in annual installments of 25,000 at the beginning of each year for 11 years and a final payment of 16,265 at the beginning of the 12th year.

Alternatively, Kimberley may receive annual installments of 13,000 at the beginning of each year for life, with a certain period of 10 years.

Calculate the present value of a 10-year deferred life annuity-due of one per annum at Kimberley's issue age. [SOA 11/89 #17]

(A) 8.5 (B) 9.0 (C) 9.5 (D) 10.0 (E) 10.5

9. You are given:

(i) X is the current value at the end of year two of a 20-year annuity-due of 1 per annum.

(ii) The annual effective interest rate for year t is $\frac{1}{8+t}$. Calculate X. [SOA 5/89 #6]

(A) $\sum_{t=9}^{28} \frac{10}{t}$ (B) $\sum_{t=9}^{28} \frac{11}{t}$ (C) $\sum_{t=10}^{29} \frac{10}{t}$ (D) $\sum_{t=10}^{29} \frac{11}{t}$ (E) $\sum_{t=11}^{30} \frac{10}{t}$

10. Determine an expression for $\frac{a_{\overline{5}|}}{a_{\overline{6}|}}$. [SOA 11/88 #7]

(A) $\dfrac{a_{\overline{2}|}+a_{\overline{3}|}}{2a_{\overline{3}|}}$ (B) $\dfrac{a_{\overline{2}|}+s_{\overline{3}|}}{1+a_{\overline{3}|}+s_{\overline{2}|}}$ (C) $\dfrac{a_{\overline{2}|}+s_{\overline{3}|}}{a_{\overline{3}|}+s_{\overline{3}|}}$ (D) $\dfrac{1+a_{\overline{2}|}+s_{\overline{2}|}}{a_{\overline{3}|}+s_{\overline{3}|}}$ (E) $\dfrac{1+a_{\overline{2}|}+s_{\overline{2}|}}{1+a_{\overline{3}|}+s_{\overline{2}|}}$

11. On January 1, an insurance company has 100,000 which is due to Linden as a life insurance death benefit. He chooses to receive the benefit annually over a period of 15 years, with the first payment immediately. The benefit he receives is based on an effective interest rate of 4% per annum.

The insurance company earns interest at an effective rate of 5% per annum. Every July 1, the company pays 100 in expenses and taxes to maintain the policy.

At the end of nine years, the company has X remaining.

Calculate X. [SOA 11/88 #16]

(A) 46,000 (B) 47,100 (C) 47,700 (D) 52,800 (E) 53,900

12. John took out a 2,000,000 construction loan, disbursed to him in three installments. The first installment of 1,000,000 is disbursed immediately and this is followed by two 500,000 installments at six month intervals.

The interest on the loan is calculated at a rate of 15% convertible semiannually and accumulated to the end of the second year. At that time, the loan and accumulated interest will be replaced by a 30-year mortgage at 12% convertible monthly.

The amount of the monthly mortgage payment for the first five years will be one-half of the payment for the sixth and later years. The first monthly mortgage payment is due exactly two years after the initial disbursement of the construction loan.

Calculate the amount of the 12th mortgage payment. [SOA 5/88 #12]

(A) 13,225 (B) 13,357 (C) 16,787 (D) 16,955 (E) 25,811

13. The proceeds from a life insurance policy are left on deposit, with interest credited at the end of each year. The beneficiary makes withdrawals from the fund at the end of each year t, for $t = 1, 2, \ldots, 10$.

At the minimum interest rate of 3% guaranteed in the policy, the equal annual withdrawal would be 1,000. However, the insurer credits interest at the rate of 4% for the first four years and 5% for the next six years. The actual amount withdrawn at the end of year t is

$$W_t = \frac{F_t}{\ddot{a}_{\overline{11-t}|.03}}$$

where F_t is the amount of the fund, including interest, prior to the withdrawal.

Calculate W_{10}. [SOA 11/87 #11]

(A) 1,160 (B) 1,167 (C) 1,172 (D) 1,177 (E) 1,183

14. Which of the following are equal to 1?

 I. $\frac{a_{\overline{10}|}(1+is_{\overline{10}|})}{1+\ddot{s}_{\overline{9}|}}$

 II. $v^{10}\ddot{s}_{\overline{10}|} - a_{\overline{9}|}$

 III. $(1+i)^{10}a_{\overline{10}|} - \ddot{s}_{\overline{9}|}$

 [SOA 5/87 #17]

 (A) I and II only

 (B) I and III only

 (C) II and III only

 (D) I, II, and III

 (E) The correct answer is not given by (A), (B), (C), or (D).

15. A company has a lease expiring on December 31, 1986. The company is notified that the monthly rent will double as of January 1, 1987. This rate will be good for two years. The company wishes to dampen the effect of the rent increase by paying a higher rent for 2 1/2 years, starting July 1, 1986.

 Calculate the percentage increase on July 1, 1986 assuming an interest rate of 12% compounded monthly. [SOA 11/86 #1]

 (A) 70% (B) 72% (C) 74% (D) 76% (E) 78%

16. A person age 40 wishes to accumulate a fund for retirement by depositing an amount X at the end of each year into an account paying 4% interest. At age 65, the person will use the entire account balance to purchase a 15-year 5% annuity-immediate with annual payments of \$10,000.

 Find X. [SOA SAMPLE/84 #1]

 (A) \$2,490 (B) \$2,520 (C) \$2,550 (D) \$2,580 (E) \$2,610

17. At the beginning of each year for ten years \$100 is deposited into a savings account.

 At a simple annual interest rate of i%, the total amount in the account is \$1,275 at the end of ten years.

 To the nearest \$5, what would be the total amount in the account at the end of ten years if interest had been compounded at an effective annual interest rate of i%. [CAS 11/82 #1]

 (A) \$1,315 (B) \$1,320 (C) \$1,325 (D) \$1,330 (E) \$1,335

18. An annuity provides a payment of \$n at the end of each year for n years.

 The effective annual interest rate is $1/n$.

 What is the present value of the annuity? [CAS 11/82 #7]

 (A) $n^2\left[1-\left(\frac{n}{n+1}\right)^n\right]$

 (B) $n^2\left[1+\left(\frac{n}{n+1}\right)^n\right]$

 (C) $n^2 - n^{n+1}(n+1)^{-n+2}$

 (D) $n^2(n+1)^{-n}$

 (E) $n^2 - n^{n+1}(n+1)^{-n+1}$

Solutions to Past Exam Questions on Sections 3a to 3c

1. If you are puzzled by the statement that "Each 1000 will provide for 9.65 of income at the beginning of each month starting on his 65th birthday until the end of his life," you are not alone. Many students don't know what this means. The statement refers to what is known as a ***life annuity***, i.e., an annuity that is paid to someone for as long as he or she lives, rather than for a fixed or certain number of periods. (Obviously, the PV of a life annuity depends on mortality rates as well as the interest rate.)

   Life annuities are covered in later actuarial exams, so it might be considered unfair to include this question in an FM/2 exam. However, there is actually enough information given in the question, if properly interpreted, for an FM/2 student to answer it.

   A life annuity is a product sold by life insurance companies. What the statement means is this: If the man pays the insurance company \$1,000 on his 65th birthday, the company will guarantee to pay him \$9.65 at the beginning of each month for as long as he lives. In this problem, the man wants to receive \$3,000 per month for life starting at age 65. If \$1,000 would guarantee \$9.65 per month, how much does he have to pay the company to guarantee \$3,000 per month?

   One way to look at this is to note that if \$1,000 guarantees \$9.65 per month, then $\frac{\$1{,}000}{9.65}$ guarantees \$1.00 per month. So to guarantee \$3,000 per month, we multiply this by 3,000:

   $$\text{Amount to be paid at age 65 to guarantee a monthly life income of \$3,000}$$
   $$= 3{,}000 \times \frac{\$1,000}{9.65}$$

   AV of contributions at age 65 $= X\ddot{s}_{\overline{300}|}$ at $\frac{2}{3}\%$.

   This must equal $(3{,}000)\frac{\$1{,}000}{9.65}$ to provide a monthly income of 3,000 for life.

   $$X = \frac{(3{,}000)\frac{\$1{,}000}{9.65}}{\ddot{s}_{\overline{300}|}} = \frac{310{,}880.8}{957.366577} = \boxed{\mathbf{324.72}} \qquad \text{ANS. (A)}$$

2. The most convenient comparison date is the end of the 40th year.

   $$\text{Susan: } 100s_{\overline{40}|} = Xa_{\overline{15}|} \text{ at } 8\%$$
   $$X = \frac{25{,}905.65}{8.559479} = 3{,}026.55$$
   $$\text{Jeff: } 100s_{\overline{40}|} = Ya_{\overline{15}|} \text{ at } 10\%$$
   $$Y = \frac{44{,}259.26}{7.606080} = 5{,}818.93$$
   $$Y - X = \boxed{\mathbf{2{,}792.38}} \qquad \text{ANS. (A)}$$

3. The amount of John's inheritance is equal to the PV of the annuity-due that he bought:

   $$\text{John's inheritance} = 2{,}500\ddot{a}_{\overline{10}|.08}$$

   Since Jeff and John shared equally in the inheritance, Jeff's share is also equal to $2{,}500\ddot{a}_{\overline{10}|.08}$.

   We are told that Jeff immediately invests his inheritance for two years at 9% effective, at which point his investment can buy a 15-year annuity-immediate at 8% effective with an annual payment of Z:

   $$2{,}500\ddot{a}_{\overline{10}|.08}(1.09)^2 = Za_{\overline{15}|.08}$$
   $$Z = \frac{21{,}525.07}{8.55948} = \boxed{\mathbf{2{,}514.76}} \qquad \text{ANS. (C)}$$

4. Paul withdraws $100\ddot{a}_{\overline{120}|.01} = 7{,}039.76$ at the end of years 6 to 20. P is the PV of these withdrawals at 8%.

   $$P = 7{,}039.76v^5a_{\overline{15}|} \text{ at } 8\% = (7{,}039.76)(.6806)(8.5595)$$
   $$= \boxed{\mathbf{41{,}009.68}} \qquad \text{ANS. (A)}$$

   Note: We will see in Section 3d that the PV at time 0 of payments of 7,039.76 at time 6 through time 20 can also be expressed as $7{,}039.76(a_{\overline{20}|} - a_{\overline{5}|})$. (Imagine that there are 20 payments and then deduct the PV of the first 5 imaginary payments.) Of course, this gives the same numerical result.

5. $14{,}113 = 1{,}000a_{\overline{4N|}}$ at 6.3%, $4N = 36, N = 9$. The PV of the first 9 payments is $1{,}000a_{\overline{9|}} = 6{,}713.76$. The PV of the third set of 9 payments is $v^{18}6{,}713.76 = 2{,}235.46$. The total $PV = 14{,}113$. Thus the percentage of the total PV that we want is:

$$\frac{6{,}713.76 + 2{,}235.46}{14{,}113} = \boxed{.63} \qquad \text{ANS. (C)}$$

6. Use the end of the 10th year as the comparison date. The AV of the payments is:

$$10{,}000(1.1)^{10} + 1{,}000s_{\overline{10|}10\%} = 41{,}874.80$$

This must be equal to the PV of the returns:

$$41{,}874.80 = Xa_{\overline{5|}10\%},\ X = \boxed{11{,}046} \qquad \text{ANS. (C)}$$

7.
$$1{,}000 = 100a_{\overline{11|}} + Rv^{12} \text{ at } 3.5\%$$
$$R = \frac{1{,}000 - 100(9.002)}{.6618} = 150.80$$
$$PV = 100a_{\overline{3|}.01} + 150.80v^4_{.01} = 100(2.941) + (150.80)(.961) = \boxed{439} \qquad \text{ANS. (D)}$$

Notes: (1) The final payment can be computed using the BA II Plus by entering 12 [N] 3.5 [I/Y] 1,000 [PV] 100 [+/−] [PMT] [CPT] [FV]. The result is 50.87, so final payment is $100 + 50.87 = 150.87$. See section 6d for more details.

(2) The question is a little ambiguous. The PV "at the beginning of the ninth year" could be interpreted to include the payment of 100 due at time 8. The official SOA solution excluded this payment.

8. (Final payments are explained in Section 6d.)

The PV of the first annuity that Kimberly could receive must be equal to the death benefit of 250,000:

$$250{,}000 = 25{,}000\ddot{a}_{\overline{11|}} + 16{,}265v^{11}$$

To solve for i using the BA II Plus, put the calculator in [BGN] mode. (This is one of those problems that make it difficult to avoid the [BGN] mode.) Then enter 11 [N] 250,000 [PV] 25,000 [+/−] [PMT] 16,265 [+/−] [FV] [CPT] [I/Y]. (These keystrokes compute the interest rate for which the PV of 11 payments of 25,000 and a final payment of 16,265 at time 11 equals 250,000.) The result is $i = 3.0004\%$.

Kimberly's alternative is to receive 13,000 at the beginning of each year for 10 years certain, followed by payments of 13,000 for life, i.e., what is called a 10-year deferred life annuity-due. Let X = PV of a 10-year deferred life annuity-due with annual payments of 1. (This is what the question is asking us to find.) Then we have:

$$13{,}000\left(\ddot{a}_{\overline{10|}} + X\right) = 250{,}000$$
$$X = \frac{250{,}000}{13{,}000} - 8.786 = \boxed{10.5} \qquad \text{ANS. (E)}$$

9. From (ii) we have:

$$i_t = \frac{1}{8+t},\ 1 + i_t = \frac{9+t}{8+t}$$

"The current value at the end of year two" of a 20-year annuity-due is the sum of the AV of the payments at time 0, 1, and 2, and the PV of the payments at time $3, 4, \ldots, 19$. Since the effective interest rate changes each year, we write the current value as follows:

$$\begin{aligned} X &= (1+i_1)(1+i_2) + (1+i_2) + 1 \\ &\quad + \frac{1}{1+i_3} + \frac{1}{(1+i_3)(1+i_4)} + \cdots + \frac{1}{(1+i_3)\ldots(1+i_{19})} \\ &= \left(\frac{10}{9}\right)\left(\frac{11}{10}\right) + \frac{11}{10} + 1 + \frac{11}{12} + \left(\frac{11}{12}\right)\left(\frac{12}{13}\right) \\ &\quad + \cdots + \left(\frac{11}{12}\right)\left(\frac{12}{13}\right)\cdots\left(\frac{27}{28}\right) \\ &= \frac{11}{9} + \frac{11}{10} + \frac{11}{11} + \frac{11}{12} + \frac{11}{13} + \cdots + \frac{11}{28} \\ &= \boxed{\sum_{t=9}^{28} \frac{11}{t}} \qquad \text{ANS. (B)} \end{aligned}$$

10. To get the format of the answer choices, express the ratio as follows:

$$\frac{a_{\overline{5}|}}{a_{\overline{6}|}} = \frac{a_{\overline{3}|} + v^3 a_{\overline{2}|}}{a_{\overline{3}|} + v^3 a_{\overline{3}|}}$$

Multiplying numerator and denominator by $(1+i)^3$:

$$\frac{a_{\overline{5}|}}{a_{\overline{6}|}} = \frac{(1+i)^3 a_{\overline{3}|} + a_{\overline{2}|}}{(1+i)^3 a_{\overline{3}|} + a_{\overline{3}|}}$$
$$= \boxed{\frac{s_{\overline{3}|}+a_{\overline{2}|}}{s_{\overline{3}|}+a_{\overline{3}|}}} \qquad \text{ANS. (C)}$$

11. Annual benefit $= \frac{100{,}000}{\ddot{a}_{\overline{15}|.04}} = \frac{100{,}000}{11.563} = 8{,}648.27$

$$X = 100{,}000(1.05)^9 - 8{,}648.27\ddot{s}_{\overline{9}|.05} - 100\ddot{s}_{\overline{9}|.05} v_{.05}^{\frac{1}{2}}$$
$$= 100{,}000(1.5513) - 11.578[8{,}648.27 + (100)(.9759)] = \boxed{53{,}870} \qquad \text{ANS. (E)}$$

12. If x = monthly mortgage payment in the first 5 years, we have (using the end of the 2nd year as the comparison date):

$$x\left(2\ddot{a}_{\overline{360}|.01} - \ddot{a}_{\overline{60}|.01}\right) = 1{,}000{,}000(1.075)^4$$
$$+ 500{,}000\left(1.075^3 + 1.075^2\right) = 2{,}534{,}430$$
$$x = \frac{2{,}534{,}430}{2\ddot{a}_{\overline{360}|} - \ddot{a}_{\overline{60}|}} = \frac{2{,}534{,}430}{(2)(98.190514) - 45.404589} = \boxed{16{,}787} \qquad \text{ANS. (C)}$$

13.
$$F_0 = 1{,}000a_{\overline{10}|.03}, \quad F_1 = 1{,}000a_{\overline{10}|.03}(1.04)$$
$$W_1 = \frac{1{,}000a_{\overline{10}|.03}(1.04)}{\ddot{a}_{\overline{10}|.03}} = 1{,}000\left(\frac{1.04}{1.03}\right)$$
$$F_2 = (F_1 - W_1)\,1.04 = 1{,}000\left(\frac{1.04}{1.03}\right)\left[a_{\overline{10}|}(1.03) - 1\right](1.04)$$
$$= 1{,}000a_{\overline{9}|.03}\left(\frac{1.04}{1.03}\right)(1.04)$$
$$W_2 = \frac{F_2}{\ddot{a}_{\overline{9}|.03}} = 1{,}000\left(\frac{1.04}{1.03}\right)^2$$
$$\text{Similarly, } W_{10} = \frac{1{,}000(1.04)^4(1.05)^6}{1.03^{10}} = \frac{1{,}000(1.1699)(1.3401)}{1.3439}$$
$$= \boxed{1{,}166.59} \qquad \text{ANS. (B)}$$

Note: This is a tough problem under exam conditions. It might be smart to skip it and spend time on questions that you would be more likely to succeed in doing.

14. I. Numerator: $a_{\overline{10}|}(1 + is_{\overline{10}|}) = a_{\overline{10}|}[1 + (1+i)^{10} - 1] = s_{\overline{10}|}$ and Denominator: $1 + \ddot{s}_{\overline{9}|} = s_{\overline{10}|}$

Thus $\boxed{\mathbf{I = 1}}$

II. $v^{10}\ddot{s}_{\overline{10}|} - a_{\overline{9}|} = \ddot{a}_{\overline{10}|} - a_{\overline{9}|} = 1 + a_{\overline{9}|} - a_{\overline{9}|}$. Thus $\boxed{\mathbf{II = 1}}$

III. $(1+i)^{10}a_{\overline{10}|} - \ddot{s}_{\overline{9}|} = s_{\overline{10}|} - \ddot{s}_{\overline{9}|} = s_{\overline{10}|} - (s_{\overline{10}|} - 1)$. Thus $\boxed{\mathbf{III = 1}}$

All three are equal to 1. ANS. (D)

15. Let x = monthly rent as originally scheduled for 1986, kx = rent effective 7/1/86 under the revised schedule. Using 7/1/86 as the comparison date:

$$kx\ddot{a}_{\overline{30}|} = x\ddot{a}_{\overline{6}|} + 2xv^6\ddot{a}_{\overline{24}|} \text{ at } 1\%$$
$$k = \frac{a_{\overline{6}|} + 2v^6 a_{\overline{24}|}}{a_{\overline{30}|}} = \frac{5.795 + 2(.9420)(21.243)}{25.808}$$
$$= \boxed{\mathbf{1.775}\text{, i.e., increase of } \mathbf{77.5\%}} \qquad \text{ANS. (E)}$$

16. $Xs_{\overline{25}|.04} = 10{,}000a_{\overline{15}|.05}\,, \quad X = \frac{103{,}796.58}{41.646} = \boxed{\mathbf{2{,}492}}$ ANS. (A)

17. At simple interest rate $i\%$, the AV of the 10 deposits is:

$$\begin{aligned}100\left[\left(1+\frac{i}{100}\right)+\left(1+\frac{2i}{100}\right)+\cdots+\left(1+\frac{10i}{100}\right)\right] &= 1{,}275\\ 100\left(10+\frac{55i}{100}\right) &= 1{,}275\\ i = \frac{275}{55} &= 5\end{aligned}$$

At compound interest effective rate 5%:

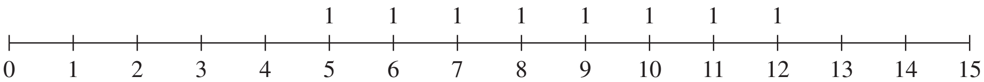
$$AV = 100\ddot{s}_{\overline{10}|.05} = \boxed{\mathbf{1{,}320.68}} \qquad \text{ANS. (B)}$$

18. $i = \frac{1}{n}$, $1+i = \frac{n+1}{n}$, $v = \frac{n}{n+1}$

$$\begin{aligned}PV &= na_{\overline{n}|} \text{ at rate } i = \frac{1}{n}\\ &= n\left(\frac{1-v^n}{i}\right) = n\frac{\left[1-\left(\frac{n}{n+1}\right)^n\right]}{\frac{1}{n}}\\ &= \boxed{\boldsymbol{n^2\left[1-\left(\frac{n}{n+1}\right)^n\right]}} \qquad \text{ANS. (A)}\end{aligned}$$

3d Deferred Annuities (or Playing "Now you see it...")

Consider the following series of payments:



Let's determine the value of these 8 payments as of various dates:

As of times 4, 5, 12, and 13, clearly the values can be expressed using symbols we defined earlier in this section:

$t = 4 \quad a_{\overline{8}|}$

$t = 5 \quad \ddot{a}_{\overline{8}|}$

$t = 12 \quad s_{\overline{8}|}$

$t = 13 \quad \ddot{s}_{\overline{8}|}$

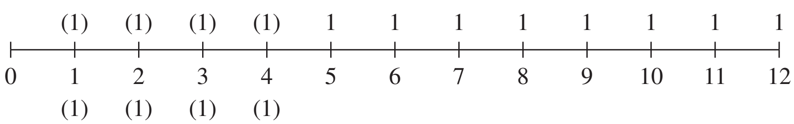
As of $t = 0$, we have a number of choices. We could discount any of the 4 values above back to time 0:

$$PV = v^4 a_{\overline{8}|} = v^5 \ddot{a}_{\overline{8}|} = v^{12} s_{\overline{8}|} = v^{13} \ddot{s}_{\overline{8}|}$$

There is a special symbol for this kind of ***deferred annuity***: ${}_{4|}a_{\overline{8}|}$. We can interpret this symbol as follows: "Go to time 4. Pay what the symbol to the right of the vertical line says—in this case, an annuity-immediate for 8 years. Thus the payments run from $t = 5$ to $t = 12$."

How would you write this PV as a deferred annuity-due? Answer: ${}_{5|}\ddot{a}_{\overline{8}|}$. ("Go to time 5. Start paying what the symbol to the right says," etc.)

We could also determine the present value by playing "Now you see it...." Place ***four*** fictional payments on the diagram and immediately withdraw them:



The PV of the payments above the time line (including the four fictitious ones) is $a_{\overline{12}|}$. But this includes the PV of the fictitious payments, which is $a_{\overline{4}|}$. Thus, the PV of this deferred annuity is $a_{\overline{12}|} - a_{\overline{4}|}$.

The two most common ways to evaluate ${}_{4|}a_{\overline{8}|}$ are:

$${}_{4|}a_{\overline{8}|} = v^4 a_{\overline{8}|} = a_{\overline{12}|} - a_{\overline{4}|}$$

To generalize the symbol for a deferred annuity, consider an m-year deferred n-payment annuity-immediate. (Note that the first payment is at time $m+1$.) This annuity could also be described as an $(m+1)$-year deferred n-payment annuity-due. We have:

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= {}_{m+1|}\ddot{a}_{\overline{n}|} = v^m a_{\overline{n}|} = v^{m+1} \ddot{a}_{\overline{n}|} \\ &= a_{\overline{m+n}|} - a_{\overline{m}|} \end{aligned}$$

Consider the AV of the payments as of time 15. The two most common methods for evaluating this AV are:

$$AV = (1+i)^3 s_{\overline{8}|} = s_{\overline{11}|} - s_{\overline{3}|}$$

where the second method is based on playing "now you see it...."

There is no special symbol for this AV.

Now consider the value of the same payments as of time 7. We could express it as any of the previous values accumulated or discounted to time 6. Perhaps the simplest expression is:

$$s_{\overline{3}|} + a_{\overline{5}|}$$

This value could also be expressed as $(1+i)^3 a_{\overline{8}|}$ and many other forms.

Stepping Stones

Example 3.12

Evaluate the following at $i = 3.25\%$:

$$_{10|}a_{\overline{20}|} = ?$$
$$_{20|}\ddot{a}_{\overline{25}|} = ?$$

Solution

For $_{10|}a_{\overline{20}|}$, use the “Now you see it...” approach: $_{10|}a_{\overline{20}|} = a_{\overline{30}|} - a_{\overline{10}|}$. First, determine $a_{\overline{10}|}$, store it, then determine $a_{\overline{30}|}$, recall $a_{\overline{10}|}$ and subtract it:

10 N 3.25 I/Y 1 +/− PMT CPT PV STO 0 $(a_{\overline{10}|} = 8.422395)$

We determine $a_{\overline{30}|}$ simply by changing N to 30: 30 N CPT PV $(a_{\overline{30}|} = 18.981917)$

Press the minus key, then RCL 0 =. The answer is **10.559522**.

For $_{20|}\ddot{a}_{\overline{25}|}$, to avoid using the BGN mode, we can write $_{20|}\ddot{a}_{\overline{25}|} = {}_{19|}a_{\overline{25}|} = a_{\overline{44}|} - a_{\overline{19}|}$. Using similar keystrokes as before, you should get **9.224598**.

Example 3.13

A deposit of $1,500 is made on January 1 of every year from 2010 to 2021, inclusive. Determine the accumulated value of these deposits on January 1, 2030 if the effective rate of interest is 1.75% per annum.

Solution

There are 12 deposits (not 11). The comparison date is 9 years after the last deposit.

$$AV = s_{\overline{21}|} - s_{\overline{9}|}$$

Using keystrokes similar to Example 3.12, but with −1,500 in PMT and the FV key instead of the PV key, you should get $AV =$ **23,189.97**.

Example 3.14

Which of the following is *not* a correct expression for the present value, on January 1, 2018, of 1 paid every January 1 from 2025 to 2035, inclusive?

(A) $_{6|}a_{\overline{11}|}$ (B) $\ddot{a}_{\overline{18}|} - \ddot{a}_{\overline{7}|}$ (C) $_{7|}\ddot{a}_{\overline{11}|}$ (D) $a_{\overline{17}|} - a_{\overline{6}|}$ (E) $v^7 a_{\overline{11}|}$

Solution

(A), (B), (C), and (D) are all correct expressions for this PV. (E) is incorrect: It should be $\boldsymbol{v^6 a_{\overline{11}|}}$.

Example 3.15

You are given that:

(i) $a_{\overline{n}|} = 10$

(ii) The present value of an n-year deferred, n-year annuity-immediate is 5.

Determine i.

Solution

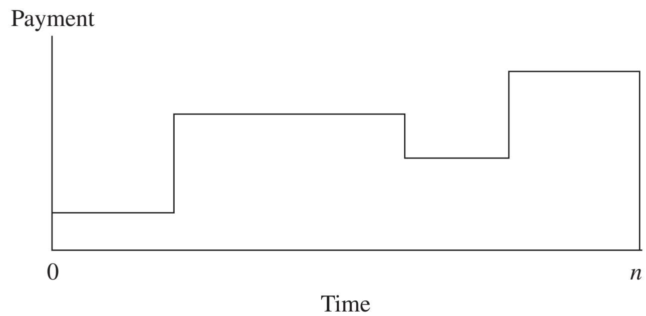
From (ii), $v^n a_{\overline{n}|} = 5$. Since $a_{\overline{n}|} = 10$, we have $v^n = 5/10 = 0.5$. Now $a_{\overline{n}|} = (1 - v^n)/i$. Substitute 0.5 for v^n:

$$a_{\overline{n}|} = 10 = (1 - v^n)/i = (1 - 0.5)/i$$
$$10i = 1 - 0.5 = 0.5$$
$$i = \boxed{\mathbf{5\%}}$$

3e A Short-Cut Method for Annuities with "Block" Payments

Shortcut Alert!

This section shows how to rapidly write down the value of an ***annuity with "block" payments***; i.e., an annuity which has payments which could be sketched this way:



In other words, payments are level for a period of years, then change to another level for another period of years, etc.

For example, consider the following annuity with block payments: \$5 for the first 8 years, \$12 for the next 7 years, \$10 for the next 7 years and \$15 for the next 6 years:

Payment:		5	…	5	12	…	12	10	…	10	15	…	15
Time:	0	1		8	9		15	16		22	23		28

We will learn to immediately write down the PV (or AV) of an annuity like this directly from the payments.

The Long Way

Before we learn the shortcut method, let's do it by a more deliberate and slower method. We will write the PV "block-by-block."

PV of first 8 payments:	$5a_{\overline{8}\vert}$
PV of next 7 payments:	$12(a_{\overline{15}\vert} - a_{\overline{8}\vert})$
PV of next 7 payments:	$10(a_{\overline{22}\vert} - a_{\overline{15}\vert})$
PV of next 6 payments:	$15(a_{\overline{28}\vert} - a_{\overline{22}\vert})$

Adding up all the present values, combining terms and writing in descending order of the periods:

$$PV = 15a_{\overline{28}|} - 5a_{\overline{22}|} + 2a_{\overline{15}|} - 7a_{\overline{8}|}$$

The Short Way

The ***rules*** are: (1) Start with the payment furthest from the comparison date; (2) make adjustments (plus or minus) as you move in closer to the comparison date.

(The following description may seem longer than the "long way" but the fact is that once you practice a little, you can look at the block payments and write the PV in simplest form in 10 to 15 seconds.)

We want the PV, so the comparison date is time 0. We start with the furthest payment (\$15 at time 28) and immediately write $15a_{\overline{28}|}$. We move in closer from time 28 toward time 0 until there is a change. This occurs at time 22, when payments *decrease by \$5* (from \$15 to \$10), so we write $-5a_{\overline{22}|}$. We move in closer, see another change at $t = 15$, when payments *increase by \$2* (from \$10 to \$12) so we write $+2a_{\overline{15}|}$. Finally, the last change is at time 8, a *decrease of \$7* (from \$12 to \$5), so we write $-7a_{\overline{8}|}$.

Putting all of this together, we have:

$$PV = 15a_{\overline{28}|} - 5a_{\overline{22}|} + 2a_{\overline{15}|} - 7a_{\overline{8}|}$$

What we have just done is to start with a level annuity of \$15 per year and then make successive adjustments to the \$15 payments to get the payments we want.

Note that this is identical to the answer obtained by "the long way."

Stepping Stones

Example 3.16

Write down directly in simplest annuity form the PV of the following payments:

Time	Payment
1 to 10	\$5
11 to 18	\$8
19 to 23	\$12
24 to 30	\$20

Solution

Starting with the payment of \$20 at time 30 and making the successive adjustments as we move closer in to time 0, we obtain

$$\boldsymbol{PV = 20a_{\overline{30}|} - 8a_{\overline{23}|} - 4a_{\overline{18}|} - 3a_{\overline{10}|}}$$

Accumulated Value

The AV of annuities with block payments is obtained in much the same way as the PV. For example, consider the annuity just above. The comparison date is time 30 if we want the AV on the date of the last payment, so we start with the *furthest payment* from time 30, which is \$5 at time 1. We immediately write $5s_{\overline{30}|}$. As we move toward the comparison date of time 30, we see that the first change is an *increase of \$3* (from \$5 to \$8) at time 11, so our adjustment term is $+3s_{\overline{20}|}$. (20 is the number of payments that we must increase by \$3, i.e., the payments from time 11 to time 30, inclusive, or 30−10 payments.) The next change is an *increase of \$4* (from \$8 to \$12) at time 19, so the adjustment term is $+4s_{\overline{12}|}$. Finally, payments *increase by \$8* (from \$12 to \$20) at time 24, so the adjustment term is $+8s_{\overline{7}|}$. Putting it all together:

$$AV = 5s_{\overline{30}|} + 3s_{\overline{20}|} + 4s_{\overline{12}|} + 8s_{\overline{7}|}$$

Stepping Stones

Example 3.17

Write the present value at time 0 and the accumulated value on the date of the last payment for the following annuities:

Annuity #1		Annuity #2	
Time	**Payment**	**Time**	**Payment**
1 to 5	5	1 to 10	8
6 to 12	10	11 to 15	0
13 to 18	5	16 to 22	10
19 to 27	10	23 to 25	12

Solution

Annuity #1:

$$\boldsymbol{PV = 10a_{\overline{27}|} - 5a_{\overline{18}|} + 5a_{\overline{12}|} - 5a_{\overline{5}|}}$$

$$\boldsymbol{AV = 5s_{\overline{27}|} + 5s_{\overline{22}|} - 5s_{\overline{15}|} + 5s_{\overline{9}|}}$$

Annuity #2:

$$\boldsymbol{PV = 12a_{\overline{25}|} - 2a_{\overline{22}|} - 10a_{\overline{15}|} + 8a_{\overline{10}|}}$$

$$\boldsymbol{AV = 8s_{\overline{25}|} - 8s_{\overline{15}|} + 10s_{\overline{10}|} + 2s_{\overline{3}|}}$$

Example 3.18

Two annuities have the following payments:

	Payment	
Time	**Annuity #1**	**Annuity #2**
1 to 5	5	0
6 to 10	0	5
11 to 15	5	0
16 to 20	0	5

The present value of Annuity #1 is 42.127. The present value of Annuity #2 is 37.692. Determine i.

Solution

$$\text{PV of \#1} = 5a_{\overline{15}|} - 5a_{\overline{10}|} + 5a_{\overline{5}|} = 42.127$$
$$\text{PV of \#2} = 5a_{\overline{20}|} - 5a_{\overline{15}|} + 5a_{\overline{10}|} - 5a_{\overline{5}|} = 37.692$$

If we add these two PVs, we get:

$$5a_{\overline{20}|} = 79.819 \quad \text{and} \quad a_{\overline{20}|} = 15.9638$$

Using the calculator to determine i, we get $i =$ **2.25%** (to 2 decimals).

Example 3.19

Which of the following is a correct expression for the accumulated value, on the date of the last payment, of 3 paid in years 1 to 7, 8 paid in years 8 to 12 and 5 paid in years 13 to 18?

(A) $3s_{\overline{18}|} + 5s_{\overline{10}|} - 3s_{\overline{5}|}$

(B) $3s_{\overline{7}|}(1+i)^{10} + 8s_{\overline{5}|}(1+i)^5 + 5s_{\overline{5}|}$

(C) $3s_{\overline{18}|} + 8s_{\overline{11}|} + 5s_{\overline{6}|}$

(D) $3s_{\overline{18}|} + 5s_{\overline{11}|} - 3s_{\overline{5}|}$

(E) $3s_{\overline{7}|}(1+i)^{11} + 8s_{\overline{5}|}(1+i)^6 + 5s_{\overline{6}|}$

Solution

ANS. (E). Using the "block payment" approach, a correct answer would also be $3s_{\overline{18}|} + 5s_{\overline{11}|} - 3s_{\overline{6}|}$.

3f Perpetuities

Perpetuity-Immediate

Consider an annuity-immediate with annual payments of \$1.00 where the payments continue forever. Such an annuity is called a ***perpetuity-immediate***. At first glance, it might appear that the PV of this annuity is infinite. But at positive rates of interest, the PV has a limit:

$$\lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \left(\frac{1-v^n}{i}\right) = \frac{1}{i}$$

since $v < 1$ and $\lim_{n\to\infty} v^n = 0$ when $i > 0$.

The $\lim_{n\to\infty} a_{\overline{n}|}$ is usually written as $a_{\overline{\infty}|}$.

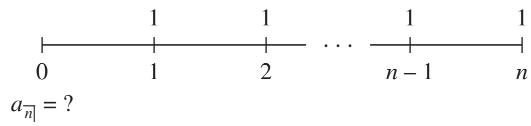
We could also have determined the limit by summing the infinite geometric progression:

$$\begin{aligned} a_{\overline{\infty}|} &= v + v^2 + \cdots \\ &= \frac{v}{1-v} = \frac{v}{d} = \frac{v}{iv} = \frac{1}{i} \end{aligned}$$

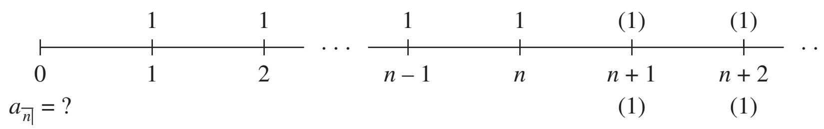
Still another way to determine the limit is to use a verbal explanation ("general reasoning"), as follows: I deposit \$1.00 in a bank crediting interest at i effective. It is clear that I could withdraw i at the end of each year forever and always keep the original deposit intact. Thus, \$1.00 deposited today provides for a perpetuity-immediate with annual payments of i. By proportion, a deposit of $\frac{1}{i}$ provides a perpetuity-immediate with annual payments of \$1.00, i.e.:

$$a_{\overline{\infty}|} = \frac{1}{i}$$

Incidentally, we can use the PV of a perpetuity to derive the formula $a_{\overline{n}|} = \frac{1-v^n}{i}$ by playing "Now you see it...." We would like to determine the PV of the following payments:



Let's put fictitious payments at time $(n+1), (n+2), \ldots$ forever, and immediately withdraw them:



The payments above the time line (including the fictitious payments) form a perpetuity-immediate, with PV$=\frac{1}{i}$. But the PV overstates $a_{\overline{n}|}$ by the PV of the fictitious payments, which form an n-year ***deferred perpetuity-immediate***. The PV of this deferred perpetuity is $v^n\left(\frac{1}{i}\right)$. Thus we have:

$$\begin{aligned} a_{\overline{n}|} &= \frac{1}{i} - v^n\left(\frac{1}{i}\right) \\ &= \frac{1-v^n}{i} \end{aligned}$$

Perpetuity-Due

A perpetuity-due of \$1.00 per annum begins with a payment of \$1.00 today and continues forever. Thus,

$$\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \lim_{n\to\infty}\left(\frac{1-v^n}{d}\right) = \frac{1}{d}$$

We could also sum the progression:

$$\begin{aligned} \ddot{a}_{\overline{\infty}|} &= 1 + v + v^2 + \cdots \\ &= \frac{1}{1-v} = \frac{1}{d} \end{aligned}$$

Another approach is to see that a perpetuity-due provides \$1.00 today plus exactly the same payments as a perpetuity-immediate. We have:

$$\ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|} = 1 + \frac{1}{i} = \frac{1+i}{i} = \frac{1}{d}$$

since $d = \frac{i}{1+i}$

Since $\ddot{a}_{\overline{\infty}|}$ exceeds $a_{\overline{\infty}|}$ by the first payment of \$1.00, we can put it this way:

$$\begin{aligned} \ddot{a}_{\overline{\infty}|} - a_{\overline{\infty}|} &= 1 \\ \frac{1}{d} - \frac{1}{i} &= 1 \end{aligned}$$

We covered this identity in Section 1a. Now you can give it a verbal explanation.

Stepping Stones

Example 3.20

The present value of a perpetuity-due with annual payments of 1,000 is 39,610.04. Determine i.

Solution

$$\begin{aligned} PV &= 1{,}000\ddot{a}_{\overline{\infty}|} = 1{,}000/d = 39{,}610.04 \\ d &= .025246 \\ i &= d/(1-d) = \boxed{\mathbf{2.59\%}} \quad \text{(to 2 decimals)} \end{aligned}$$

Alternatively:

$$\begin{aligned} PV &= 1{,}000(1 + a_{\overline{\infty}|}) = 1{,}000(1 + 1/i) = 39{,}610.04 \\ i &= \boxed{\mathbf{2.59\%}} \end{aligned}$$

Example 3.21

Determine the present value of a 10-year deferred perpetuity-immediate with semiannual payments of 1 at a nominal rate of interest of 1.6% compounded semiannually.

Solution

The effective semiannual rate of interest is 0.8%. You should interpret "a 10-year deferred perpetuity-immediate with semiannual payments" as meaning that the first payment is in $10\frac{1}{2}$ years, or in 21 half-year interest periods.

$$\begin{aligned} PV &= v^{20}/.008, \quad \text{where } v^{20} = 1.008^{-20} \\ PV &= \boxed{\mathbf{106.59}} \end{aligned}$$

Example 3.22

A perpetuity-immediate pays 100 in the first 10 years, 150 in the next 10 years and 200 in all subsequent years. Determine the present value of these payments at $i = 3.85\%$.

Solution

We can use the "block payment" approach with this perpetuity:

$$PV = 200/i - 50a_{\overline{20}|} - 50a_{\overline{10}|}$$

Using the calculator, you should get $\boxed{\mathbf{4{,}097.58}}$.

Example 3.23

A perpetuity-immediate of 1 per annum has the same present value as a 10-year annuity-immediate of 4 per annum, at an effective annual rate of interest i. Determine i.

Solution

$$\begin{aligned} a_{\overline{\infty}|} &= 1/i = 4a_{\overline{10}|} = 4(1 - v^{10})/i \\ 1 &= 4 - 4v^{10} \\ v^{10} &= 0.75 \\ i &= \boxed{\mathbf{2.92\%}} \quad \text{(2 decimals)} \end{aligned}$$

Summary of Concepts and Formulas in Sections 3d to 3f

1. The PV of an m-year deferred annuity-immediate with n payments is:

$$_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

2. The symbol for the deferred annuity in (1) can also be written as:

$$_{m+1|}\ddot{a}_{\overline{n}|}$$

3. The ***PV of an annuity with "block payments"*** can be written very quickly by starting with the payments farthest from the comparison date and making adjustments (plus or minus) as you move closer in. (See Section 3e for examples.)

4. (a) The PV of a perpetuity-immediate is:

$$a_{\overline{\infty}|} = \frac{1}{i}$$

   (b) The PV of a perpetuity-due is:

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$$

   (c) The PV of a perpetuity-due exceeds the PV of a perpetuity-immediate by the payment of 1 at time 0:

$$\frac{1}{d} - \frac{1}{i} = 1$$

Past Exam Questions on Sections 3d to 3f

Deferred Annuities and Annuities with Block Payments

1. To accumulate 8000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first n years and 196 at the end of each of the next $2n$ years.

   The annual effective rate of interest is i. You are given $(1+i)^n = 2.0$.

   Determine i. [11/01 #12]

   (A) 11.25% (B) 11.75% (C) 12.25% (D) 12.75% (E) 13.25%

2. Chuck needs to purchase an item in 10 years. The item costs 200 today, but its price inflates at 4% per year.

   To finance the purchase, Chuck deposits 20 into an account at the beginning of each year for 6 years. He deposits an additional X at the beginning of years 4, 5 and 6 to meet his goal.

   The annual effective interest rate is 10%.

   Calculate X. [11/00 #38]

   (A) 7.4 (B) 7.9 (C) 8.4 (D) 8.9 (E) 9.4

3. Jim began saving money for his retirement by making monthly deposits of 200 into a fund earning 6% interest compounded monthly. The first deposit occurred on January 1, 1985.

   Jim became unemployed and missed making deposits 60 through 72. He then continued making monthly deposits of 200.

   How much did Jim accumulate in his fund on December 31, 1999? [5/00 #47]

   (A) 53,572 (B) 53,715 (C) 53,840 (D) 53,966 (E) 54,184

4. Ms. Smith has two grandchildren, Adam and Evelyn. Adam will be enrolling in college on September 1, 2003, and Evelyn will be enrolling in college on September 1, 2005. Ms. Smith wishes to give both Adam and Evelyn $1,000 at the beginning of each of their four years of college.

   Ms. Smith will fund these payments by making five level annual deposits of P into an account earning an annual effective interest rate of 7%, with the first deposit on September 1, 1998.

   Determine the value of P. [CAS 5/99 #5]

   (A) Less than $1,050

   (B) At least $1,050, but less than $1,150

   (C) At least $1,150, but less than $1,250

   (D) At least $1,250, but less than $1,350

   (E) At least $1,350

5. A loan of amount $a_{\overline{10}|}$, made at time $t = 0$, is to be repaid by 10 annual payments of 1, beginning at time $t = 1$ and ending at time $t = 10$. At time $t = 4$, the borrower has financial troubles and can only pay $(1 - v^7)$. If he then returns to his original payment schedule of 1 at times $t = 5$ through $t = 9$, how much will his payment at $t = 10$ need to be in order to pay the loan off in full? [CAS 5/99 #6]

   (A) $1+v$ (B) $1+v^2$ (C) $v+v^2$ (D) $1+i$ (E) $1+i^2$

6. Consider an annuity that pays 1 at the beginning of each year for $k+m$ years.

   Which of the following expressions does *not* give the value of this annuity at the end of year k? [CAS 5/98 #10]

   (A) $a_{\overline{k+m}|}(1+i)^{k+1}$ (B) $s_{\overline{k+m}|}v^m$ (C) $s_{\overline{k+1}|} + a_{\overline{m-1}|}$ (D) $\ddot{s}_{\overline{k}|} + \ddot{a}_{\overline{m}|}$ (E) $1 + \ddot{s}_{\overline{k}|} + a_{\overline{m-1}|}$

7. Eloise plans to accumulate 100,000 at the end of 42 years. She makes the following deposits:

   (i) X at the beginning of years 1–14;

   (ii) No deposits at the beginning of years 15–32, and

   (iii) Y at the beginning of years 33–42.

   The annual effective interest rate is 7%.

$$X - Y = 100.$$

   Calculate Y. [SOA 5/98 #7]

   (A) 479 (B) 499 (C) 519 (D) 539 (E) 559

8. Jim borrowed 10,000 from Bank X at an annual effective rate of 8%. He agreed to repay the bank with five level annual installments at the end of each year.

   At the same time, he also borrowed 15,000 from Bank Y at an annual effective rate of 7.5%. He agreed to repay this loan with five level annual installments at the end of each year.

   He lent the 25,000 to Wayne immediately in exchange for four annual level repayments at the end of each year, at an annual effective rate of 8.5%.

   Jim can only reinvest the proceeds at an annual effective rate of 6%.

   Immediately after repaying the loans to the banks in full, determine how much Jim has left. [SOA 5/98 #11]

   (A) 323 (B) 348 (C) 373 (D) 398 (E) 423

9. Janet buys a $20,000 car. Prevailing market rates are nominal 8.0% annual interest, convertible monthly. The dealership offers her the choice of a rebate upon purchase of the car for cash, or alternately Janet can make no down payment and 60 monthly payments based on a nominal 2.5% annual interest, convertible monthly. The first payment would be due one month after the purchase of the car. The amount of the rebate is set so that the dealership is indifferent as to whether Janet takes the rebate or finances the car at the offered below market interest rate.

   Determine the amount of the rebate. [CAS 5/93 #13]

   (A) Less than $2,450

   (B) At least $2,450 but less than $2,550

   (C) At least $2,550 but less than $2,650

   (D) At least $2,650 but less than $2,750

   (E) At least $2,750

10. You are given:

    (i) the present value of an annuity-due that pays 300 every 6 months during the first 15 years and 200 every 6 months during the second 15 years is 6000;

    (ii) the present value of a 15-year deferred annuity-due that pays 350 every 6 months for 15 years is 4000; and

    (iii) the present value of an annuity-due that pays 100 every 6 months during the first 15 years and 200 every 6 months during the next 15 years is X.

    Determine X. [SOA 11/90 #20]

    (A) 3220 (B) 3320 (C) 3420 (D) 3520 (E) 3620

11. An annuity-immediate pays 10 at the ends of years 1 and 2, 9 at the ends of years 3 and 4, etc., with payments decreasing by 1 every second year, until nothing is paid. The effective annual rate of interest is 5%.

    Calculate the present value of this annuity-immediate. [SOA 5/90 #8]

    (A) 71 (B) 78 (C) 84 (D) 88 (E) 94

12. Tom borrows 100 at an annual effective interest rate of 4% and agrees to repay it with 30 annual installments. The amount of each payment in the last 20 years is set at twice that in the first 10 years.

At the end of 10 years, Tom has the option to repay the entire loan with a final payment X, in addition to the regular payment. This will yield the lender an annual effective rate of 4.5% over the 10-year period.

Calculate X. [SOA 5/89 #12]

(A) 89 (B) 94 (C) 99 (D) 104 (E) 109

13. Annuities X and Y provide the following payments:

End of Year	Annuity X	Annuity Y
1–10	1	K
11–20	2	0
21–30	1	K

Annuities X and Y have equal present values at an effective annual interest rate i such that $v^{10} = 1/2$.

Determine K. [SOA SAMPLE/83 #2]

(A) 4/3 (B) 3/2 (C) 5/3 (D) 7/4 (E) 9/5

Perpetuities

14. A perpetuity-immediate pays X per year. Brian receives the first n payments, Colleen receives the next n payments, and Jeff receives the remaining payments. Brian's share of the present value of the original perpetuity is 40%, and Jeff's share is K.

Calculate K. [5/01 #5]

(A) 24% (B) 28% (C) 32% (D) 36% (E) 40%

15. At an annual effective interest rate of i, $i > 0\%$, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 6, is 32.

At the same annual effective rate of i, the present value of a perpetuity-immediate paying 1 at the end of each 4-month period is X.

Calculate X. [5/01 #17]

(A) 38.8 (B) 39.8 (C) 40.8 (D) 41.8 (E) 42.8

16. The present values of the following three annuities are equal:

(i) perpetuity-immediate paying 1 each year, calculated at an annual effective interest rate of 7.25%

(ii) 50-year annuity-immediate paying 1 each year, calculated at an annual effective interest rate of $j\%$

(iii) n-year annuity-immediate paying 1 each year, calculated at an annual effective interest rate of $j - 1\%$

Calculate n. [5/01 #50]

(A) 30 (B) 33 (C) 36 (D) 39 (E) 42

17. A loan is to be repaid by annual payments continuing forever, the first one due one year after the loan is made. Find the amount of the loan if the payments are $1, 2, 3, 1, 2, 3, \dots$ assuming an annual effective interest rate of 10%. [CAS 11/99 #5]

(A) Less than 19

(B) At least 19, but less than 20

(C) At least 20, but less than 21

(D) At least 21, but less than 22

(E) At least 22

18. Mary deposits 1000 into a fund at the beginning of each year for 10 years. At the end of 15 years, she makes an additional deposit of X.

    At the end of 20 years, Mary uses the accumulated balance in the fund to buy a perpetuity-immediate with annual payments of 2000 per year for 10 years, and 1000 per year thereafter.

    Interest is credited at an annual effective rate of 5%.

    Calculate X. [SOA 5/95 #4]

    (A) 4865 (B) 5065 (C) 5265 (D) 5465 (E) 5665

19. The University of the State of Turmoil wishes to invest $100,000 in an interest bearing account. Beginning 6 years after the deposit, scholarship payments of $10,000 per year are to be made in perpetuity. Determine the minimum effective annual rate of interest that the University must earn on its investments in order to fund this perpetuity as planned. [CAS 5/94 #9]

    (A) Less than 7.04%

    (B) At least 7.04% but less than 7.08%

    (C) At least 7.08% but less than 7.12%

    (D) At least 7.12% but less than 7.16%

    (E) At least 7.16%

20. Mark receives 500,000 at his retirement. He invests $500{,}000 - X$ in an annual payment 10-year annuity-immediate and X in an annual payment perpetuity-immediate.

    His total annual payments received during the first 10 years are twice as large as those received thereafter.

    The annual effective rate of interest is 6%.

    Calculate X. [SOA 11/93 #5]

    (A) 345,835 (B) 346,335 (C) 346,835 (D) 347,335 (E) 348,835

21. Ralph buys a perpetuity-due paying 500 annually. He deposits the payments into a savings account earning interest at an effective annual rate of 10%.

    Ten years later, before receiving the eleventh payment, Ralph sells the perpetuity based on an effective annual interest rate of 10%.

    Using the proceeds from the sale plus the money in the savings account, Ralph purchases an annuity due paying X per year for 20 years at an effective annual rate of 10%.

    Calculate X. [SOA 11/92 #4]

    (A) 1145 (B) 1260 (C) 1385 (D) 1525 (E) 1675

22. At a nominal rate of interest i, convertible semiannually, the present value of a series of payments of 1 at the end of every 2 years, forever, is 5.89.

    Calculate i. [SOA 5/91 #1]

    (A) 6% (B) 7% (C) 8% (D) 9% (E) 10%

23. The following three series of payments have the same present value of P:

    (i) a perpetuity-immediate of 2 per year at an annual effective interest rate of i;

    (ii) a 20-year annuity-immediate of X per year at an annual effective interest rate of $2i$; and

    (iii) a 20-year annuity-due of $0.96154X$ per year at an annual effective interest rate of $2i$.

    Calculate P. [SOA 5/91 #7]

    (A) 80 (B) 85 (C) 90 (D) 95 (E) 100

24. Deposits of 1000 are placed into a fund at the beginning of each year for 30 years.

At the end of the 40th year, annual payments commence and continue forever.

Interest is at an effective annual rate of 5%.

Calculate the annual payment. [SOA 5/91 #10]

(A) 5440 (B) 5430 (C) 5420 (D) 5410 (E) 5400

25. The death benefit on a life insurance policy can be paid in any of the following ways, each of which has the same present value as the death benefit:

(i) a perpetuity of 120 at the end of each month;

(ii) 365.47 at the end of each month for n years; and

(iii) a payment of 17866.32 at the end of n years.

Calculate the amount of the death benefit. [SOA 5/91 #17]

(A) 8000 (B) 9000 (C) 10000 (D) 12000 (E) 15000

26. Victor wants to purchase a perpetuity paying 100 per year with the first payment due at the end of year 11. He can purchase it by either:

(i) paying 90 per year at the end of each year for 10 years; or

(ii) paying K per year at the end of each year for the first 5 years and nothing for the next 5 years.

Calculate K. [SOA 11/90 #8]

(A) 150 (B) 160 (C) 170 (D) 175 (E) 180

27. The present value of a series of payments of 2 at the end of every eight years, forever, is equal to 5.

Calculate the effective rate of interest. [SOA 11/89 #4]

(A) 0.023 (B) 0.033 (C) 0.040 (D) 0.043 (E) 0.052

28. A perpetuity pays 1 at the end of every year plus an additional 1 at the end of every second year.

The present value of the perpetuity is K for $i > 0$.

Determine K. [SOA 5/86 #9]

(A) $\dfrac{i+3}{i(i+2)}$ (B) $\dfrac{i+2}{i(i+1)}$ (C) $\dfrac{i+1}{i^2}$ (D) $\dfrac{3}{2i}$ (E) $\dfrac{i+1}{i(i+2)}$

29. A perpetuity provides the following payments:

(I) Level payments of \$1 at the end of each year for the first $2n$ years.

(II) Level payments of \$3 at the end of each year thereafter.

At the beginning of year 1 the present value of the payments specified in I equals the present value of the payments specified in II.

The effective annual interest rate is i.

What is $(1+i)^n$? [CAS 5/83 #1]

(A) 4/3 (B) 2 (C) 3/2 (D) 3 (E) 4

Solutions to Past Exam Questions on Sections 3d to 3f

1. Using the "block payments" approach:

$$98s_{\overline{3n}|} + 98s_{\overline{2n}|} = 8{,}000$$
$$\frac{(1+i)^{3n}-1}{i} + \frac{(1+i)^{2n}-1}{i} = 81.63$$

We are given that $(1+i)^n = 2$:

$$\frac{8-1}{i} + \frac{4-1}{i} = 81.63, \ i = \boxed{\textbf{12.25\%}} \qquad \text{ANS. (C)}$$

2. The AV of Chuck's deposits at time 6 is $20\ddot{s}_{\overline{6}|} + X\ddot{s}_{\overline{3}|}$. The price of the item 10 years from now is $200(1.04)^{10}$. Using time 10 as the comparison date:

$$\left(20\ddot{s}_{\overline{6}|} + X\ddot{s}_{\overline{3}|}\right)(1.10)^4 = 200(1.04)^{10}$$
$$169.74 + X(3.641) = 202.20535$$
$$3.641X = 32.46, \ X = \boxed{\textbf{8.92}} \qquad \text{ANS. (D)}$$

3. In solving this problem, we have to carefully count the number of monthly periods using the given dates. One way to do this is as follows.

   Let the first monthly deposit on 1/1/1985 be numbered as being made at $t = 1$. The deposits are made over a 15-year period, from 1985 through 1999 inclusive. (This includes the period of the missed deposits.) Under this numbering scheme, you will find that the date 12/31/1999 is at $t = 181$. (Note that the date 1/1/2000, which is one day later, is also at $t = 181$.) We want the AV at $t = 181$, just before the deposit on 1/1/2000 is made.

   As of t = 181, the first 59 deposits have an AV of $200s_{\overline{59}|}(1.005)^{122}$ and the AV of the 108 deposits made at $t = 73$ through $t = 180$ inclusive is $200\ddot{s}_{\overline{108}|} = 200s_{\overline{108}|}(1.005)$. The sum of these AVs is $\boxed{\textbf{53,839.83}}$. ANS. (C)

4. The deposits are made on 9/1/98 to 9/1/02, inclusive, so the first withdrawal for Adam will be one year after the last deposit. The first withdrawal for Evelyn will be 2 years after that. Using 9/1/02 as the comparison date:

$$Ps_{\overline{5}|} = 1{,}000a_{\overline{4}|}\left(1+v^2\right)$$
$$P = \frac{1{,}000a_{\overline{4}|}\left(1+v^2\right)}{s_{\overline{5}|}} = \frac{(3{,}387.21)(1.873439)}{5.750739}$$
$$= \boxed{\textbf{1,103.46}} \qquad \text{ANS. (B)}$$

5. The simplest approach is to note that the borrower underpaid by v^7 at time 4. He must pay the AV of v^7 at time 10 (6 years later), in addition to the regular payment of 1. Thus, his total payment at time $10 = v^7(1+i)^6 + 1 = \boxed{\boldsymbol{v+1}}$ ANS. (A)

6. At time 0, the value of the annuity is $\ddot{a}_{\overline{k+m}|}$. At time k (k years later), its value is $\ddot{a}_{\overline{k+m}|}(1+i)^k$. Since $\ddot{a}_{\overline{k+m}|} = (1+i)a_{\overline{k+m}|}$, we see immediately that (A) *does* give the correct value.

   To check (B), we have $a_{\overline{k+m}|}(1+i)^{k+1} = a_{\overline{k+m}|}(1+i)^{k+m}(1+i)^{-(m-1)} = s_{\overline{k+m}|}v^{m-1}$. $\therefore$ **(B) is incorrect.** There is no need to check the other choices. ANS. (B)

7. Using the end of 42 years as the comparison date:

$$X\ddot{s}_{\overline{14}|}(1.07)^{28} + Y\ddot{s}_{\overline{10}|} = 100{,}000$$

   Since $X - Y = 100$, we substitute $X = 100 + Y$:

$$(100+Y)(160.42997) + Y(14.78360) = 100{,}000$$
$$Y = \boxed{\textbf{479.17}} \qquad \text{ANS. (A)}$$

8.
$$\text{Bank X: } 10{,}000 = Xa_{\overline{5}|.08},\ X = 2{,}504.56$$
$$\text{Bank Y: } 15{,}000 = Ya_{\overline{5}|.075},\ Y = 3{,}707.47$$
$$\text{Total} = 6{,}212.03$$

$$\begin{aligned}\text{Wayne: } 25{,}000 = Wa_{\overline{4}|.085},\ W &= 7{,}632.20\\ \text{Difference to invest for 4 years } &= 1{,}420.17\\ AV \text{ of investment at end of 5 years } &= 1{,}420.17\ddot{s}_{\overline{4}|.06}\\ &= 6{,}585.46\end{aligned}$$

Jim has (6,585.46 − 6,212.03) left = **373** ANS. (C)

9. Let X = amount of rebate. Using the rebate option, Janet would pay $(20{,}000 - X)$ now. Using the low interest rate option, Janet would make monthly payments of $\frac{20{,}000}{a_{\overline{60}|j}}$, where $j = \frac{2.5\%}{12}$.

For the dealership, the PV of these monthly payments at the market rate is $\frac{20{,}000}{a_{\overline{60}|\frac{2.5\%}{12}}} a_{\overline{60}|\frac{8\%}{12}}$. Thus:

$$20{,}000 - X = \frac{20{,}000}{a_{\overline{60}|\frac{2.5\%}{12}}} a_{\overline{60}|\frac{8\%}{12}} = \frac{(20{,}000)(49.318433)}{56.346404}$$

and $X =$ **2,494.56** ANS. (B)

10.
$$\text{(i): } 6{,}000 = 200\ddot{a}_{\overline{60}|} + 100\ddot{a}_{\overline{30}|} \text{ at effective rate } j \text{ for } \frac{1}{2}\text{-year}$$
$$\text{(ii): } 4{,}000 = 350\left(\ddot{a}_{\overline{60}|} - \ddot{a}_{\overline{30}|}\right)$$
$$\text{(iii): } X = 200\ddot{a}_{\overline{60}|} - 100\ddot{a}_{\overline{30}|}$$

(i) and (ii) are simultaneous linear equations in $\ddot{a}_{\overline{30}|}$ and $\ddot{a}_{\overline{60}|}$. Solving:

$$\ddot{a}_{\overline{30}|} = 12.380952,\ \ddot{a}_{\overline{60}|} = 23.809524,$$

$$X = 200(23.809524) - 100(12.380952)$$
$= $ **3,523.81** ANS. (D)

Note: This problem is actually defective, although we don't believe it was ever designated as such by the SOA.

The examiners wanted you to assume that the same interest rate applied to all 3 statements (i), (ii) and (iii). But this is inconsistent with the given information. Statement (i) is true for an effective semiannual rate of about 4.517%. (You can check this out on the calculator.) Statement (ii) is true for an effective semiannual interest rate of about 2.234%. Thus, we cannot treat these two equations as if they were simultaneous linear equations. And it's meaningless to ask for the PV in (iii), since we don't know what interest rate to use.

11. Using the "block payments" approach:

$$\begin{aligned}PV &= a_{\overline{20}|} + a_{\overline{18}|} + \cdots + a_{\overline{2}|}\\ &= \frac{10 - \left(v^2 + v^4 + \cdots + v^{20}\right)}{i}\end{aligned}$$

The series in parenthesis can be summed (geometric progression), or you may notice that this is the present value of an annuity of 1 payable every other year over 20 years, i.e., $\frac{a_{\overline{20}|}}{s_{\overline{2}|}}$. Either way:

$$PV = \frac{10 - \frac{a_{\overline{20}|}}{s_{\overline{2}|}}}{.05} = \mathbf{78.42}$$ ANS. (B)

12. Let $R =$ payment in first 10 years, $2R =$ payment in next 20 years.

$$100 = 2Ra_{\overline{30}|} - Ra_{\overline{10}|} \text{ at } 4\%$$
$$R = \frac{100}{2a_{\overline{30}|} - a_{\overline{10}|}} = \frac{100}{34.5841 - 8.11090} = 3.7774$$

For the lender's yield rate to be 4.5%, we have:

$$100 = 3.7774a_{\overline{10}|.045} + Xv_{.045}^{10}$$
$$X = \left(100 - 3.7774a_{\overline{10}|.045}\right)\left(1.045^{10}\right)$$
$$= \boxed{\mathbf{108.88}} \qquad \text{ANS. (E)}$$

13. Annuity X: $PV = a_{\overline{30}|} + a_{\overline{20}|} - a_{\overline{10}|}$

Annuity Y: $PV = Ka_{\overline{30}|} - Ka_{\overline{20}|} + Ka_{\overline{10}|}$

$$K\left(a_{\overline{30}|} - a_{\overline{20}|} + a_{\overline{10}|}\right) = a_{\overline{30}|} + a_{\overline{20}|} - a_{\overline{10}|}$$
$$K = \frac{a_{\overline{30}|} + a_{\overline{20}|} - a_{\overline{10}|}}{a_{\overline{30}|} - a_{\overline{20}|} + a_{\overline{10}|}}$$
$$= \frac{\left(1 - v^{30}\right) + \left(1 - v^{20}\right) - \left(1 - v^{10}\right)}{\left(1 - v^{30}\right) - \left(1 - v^{20}\right) + \left(1 - v^{10}\right)} = \frac{1 - \left(\frac{1}{2}\right)^3 + 1 - \left(\frac{1}{2}\right)^2 - \left(1 - \frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^3 - \left[1 - \left(\frac{1}{2}\right)^2\right] + \left(1 - \frac{1}{2}\right)}$$
$$\frac{9/8}{5/8} = \boxed{\mathbf{\tfrac{9}{5}}} \qquad \text{ANS. (E)}$$

14. PV of the perpetuity $= \dfrac{X}{i}$

Brian's share $= Xa_{\overline{n}|} = .4\dfrac{X}{i}$

Colleen's share $= v^n Xa_{\overline{n}|}$

Jeff's share $= v^{2n}\dfrac{X}{i}$

From Brian's share we have:

$$a_{\overline{n}|} = \frac{.4}{i} = \frac{1 - v^n}{i}$$
$$\therefore v^n = .6,\ v^{2n} = .36$$

Thus Jeff's share $= \boxed{\mathbf{.36\tfrac{X}{i}}}$ ANS. (D)

15. 1st perpetuity: $PV = 10(v^6 + v^9 + \cdots)$

$$= \frac{10v^6}{1 - v^3} = 32$$

Letting $x = v^3$, this reduces to the quadratic $5x^2 + 16x - 16 = 0$, which has the positive root $x = v^3 = .8$.

2nd perpetuity: $X = v^{\frac{1}{3}} + v^{\frac{2}{3}} + \cdots$

$$= \frac{v^{\frac{1}{3}}}{1 - v^{\frac{1}{3}}} = \frac{.8^{\frac{1}{9}}}{1 - .8^{\frac{1}{9}}} = \boxed{\mathbf{39.8}} \qquad \text{ANS. (B)}$$

16. $\dfrac{1}{.0725} = a_{\overline{50}|\,j} = a_{\overline{n}|\,j-1\%}$

$13.793103 = a_{\overline{50}|\,j}, \quad \therefore j = 7.004382\%$ and $a_{\overline{n}|6.004382\%} = 13.793103$

$n = \boxed{\mathbf{30.2}}$ ANS. (A)

17. As of the end of the 3rd year, the 3 payments 1, 2, 3 accumulate to $1.1^2 + 2(1.1) + 3 = 6.41$. So a payment of 6.41 at time 3 is equivalent to these 3 payments. Similarly, a payment of 6.41 at times 6, 9, 12, ..., covers all of the payments of the perpetuity.

$$PV = 6.41\left(v^3 + v^6 + \cdots\right) = 6.41\frac{v^3}{1 - v^3}$$
$$= \frac{(6.41)(.751315)}{1 - .751315} = \boxed{\mathbf{19.37}} \qquad \text{ANS. (B)}$$

18. Use time 20 as the comparison date:

$$1{,}000\ddot{s}_{\overline{10}|}(1.05)^{10} + X(1.05)^5 = \frac{1{,}000}{.05} + 1{,}000a_{\overline{10}|}$$
$$X = \boxed{\textbf{4,865.13}} \qquad \text{ANS. (A)}$$

19. Using time 5 as a comparison date:

$$100{,}000(1+i)^5 = \frac{10{,}000}{i}$$
$$(1+i)^5 i = 0.1$$

Trying $i = 7.04\%, 7.08\%, 7.12\%, ...$, we find that $\boxed{\mathbf{7.08\% < i < 7.12\%}}$ ANS. (C)

20. Let $2P$ = annual payment during the first 10 years and P = annual payment thereafter. Thus, the perpetuity must provide P annually and the annuity-immediate must provide P for 10 years. Since Mark invests a total of 500,000, we have:

$$500{,}000 = P\left(a_{\overline{10}|} + \frac{1}{i}\right), \quad P = 20{,}810.13$$

Since Mark invests X in the perpetuity, we have:

$$X = \frac{P}{i} = \frac{20{,}810.13}{.06} = \boxed{\textbf{346,835.50}} \qquad \text{ANS. (C)}$$

21. Ten years after buying the perpetuity-due, the AV of the first 10 payments is $500\ddot{s}_{\overline{10}|10\%} = 8{,}765.58$. Ralph sells the perpetuity-due at this point for a price of $\frac{500}{d} = \frac{500}{i}(1+i)$ at $10\% = \frac{500}{0.1}(1.1) = 5{,}500$. Thus, the sum of the money in the savings account and the proceeds from the sale is $8{,}765.58 + 5{,}500 = 14{,}265.58$. This amount is used to buy the annuity-due:

$$14{,}265.58 = X\ddot{a}_{\overline{20}|}$$
$$X = \boxed{\textbf{1,523.30}} \qquad \text{ANS. (D)}$$

22. $5.89 = \frac{1}{j}$, where j is the effective rate for a *2-year* period. Thus, $j = \frac{1}{5.89} = .169779$. Since $\frac{i}{2}$ is the effective rate for a $\frac{1}{2}$-year period, we have:

$$\left(1+\frac{i}{2}\right)^2 = (1+j)^{\frac{1}{2}} \text{ and } i = 2\left[(i+j)^{\frac{1}{4}} - 1\right] = 2\left(1.169779^{\frac{1}{4}} - 1\right) = \boxed{\textbf{.08}} \qquad \text{ANS. (C)}$$

Note: Using the symbol i for the nominal rate compounded semiannually is poor notation. The symbol should be $i^{(2)}$.

23. Setting (ii) = (iii):

$$Xa_{\overline{20}|2i} = .96154X\ddot{a}_{\overline{20}|2i}$$
$$a_{\overline{20}|2i} = .96154(1+2i)a_{\overline{20}|2i}$$
$$1 = .96154(1+2i), \; i = .02$$

From(i):

$$P = \frac{2}{.02} = \boxed{\textbf{100}} \qquad \text{ANS. (E)}$$

24. Let X = annual payment commencing at the end of the 40th year. Using time 39 as the comparison date:

$$1{,}000\ddot{s}_{\overline{30}|}(1.05)^9 = \frac{X}{.05}, \; X = \boxed{\textbf{5,411}} \qquad \text{ANS. (D)}$$

25.
$$\text{(i): } PV = \frac{120}{i}, \text{where } i \text{ is the monthly effective rate}$$
$$\text{(ii): } PV = 365.47a_{\overline{12n}|i}$$

Dividing (i) by (ii): $\frac{120}{365.47(1-v^{12n})} = 1$

$$v^{12n} = .671656$$
$$\text{(iii): } PV = 17{,}866.32v^{12n} = (17{,}866.32)(.671656) = \boxed{\textbf{12,000}} \qquad \text{ANS. (D)}$$

26.

$$90 s_{\overline{10}|} = \frac{100}{i};\ 90\left[(1+i)^{10}-1\right] = 100,$$

$$(1+i)^{10} = \frac{19}{9} \text{ and } (1+i)^5 = \frac{\sqrt{19}}{3} = 1.452966$$

$$K s_{\overline{5}|}(1+i)^5 = \frac{K\left[(1+i)^{10}-(1+i)^5\right]}{i} = \frac{100}{i}$$

$$K = \frac{100}{(1+i)^{10}-(1+i)^5} = \frac{100}{1.452966(1.452966-1)} = \boxed{\mathbf{151.94}}$$ ANS. (A)

27.

$$2\left(v^8+v^{16}+\cdots\right) = 5$$

$$\frac{2v^8}{1-v^8} = 5,\ v^8 = \frac{5}{7} \text{ and } (1+i)^8 = \frac{7}{5} = 1.4$$

$$\therefore i = 1.4^{\frac{1}{8}} - 1 = \boxed{\mathbf{4.3\%}}$$ ANS. (D)

28.

$$K = \frac{1}{i} + v^2 + v^4 + \cdots$$

$$= \frac{1}{i} + \frac{v^2}{1-v^2}$$

To get everything in terms of i, multiply the numerator and denominator of the second term by $(1+i)^2$:

$$K = \frac{1}{i} + \frac{1}{(1+i)^2-1}$$

Combine into one fraction:

$$K = \frac{(1+i)^2-1+i}{i[(1+i)^2-1]} = \frac{1+2i+i^2-1+i}{i(1+2i+i^2-1)} = \frac{3i+i^2}{i(2i+i^2)}$$

$$= \frac{3+i}{2i+i^2} = \boxed{\frac{3+i}{i(2+i)}}$$ ANS. (A)

29.

I. $PV = a_{\overline{2n}|}$

II. $PV = 3v^{2n}\left(\frac{1}{i}\right)$

$$a_{\overline{2n}|} = \frac{3v^{2n}}{i},\ \frac{1-v^{2n}}{i} = \frac{3v^{2n}}{i},\ 1-v^{2n} = 3v^{2n},$$

$$4v^{2n} = 1,\ v^{2n} = \frac{1}{4},\ v^n = \frac{1}{2}$$

$$(1+i)^n = \boxed{\mathbf{2}}$$ ANS. (B)

3g The $a_{\overline{2n}|}\,/\,a_{\overline{n}|}$ Trick (and Variations)

Trick Alert!

One type of problem that has appeared on past exams a number of times can be solved by using the quotient $a_{\overline{2n}|}/a_{\overline{n}|}$, or variations on this theme. The following example illustrates the method:

Stepping Stones

Example 3.24

A "plain vanilla" version of this type of problem is as follows: Given that $a_{\overline{n}|} = 10$ and $a_{\overline{2n}|} = 15$, determine i.

Solution

A few lines further down, we will show that $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$. Using this result, we have:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{15}{10} = 1.5 = 1 + v^n$$

This gives $v^n = 0.5$. All that remains is to plug v^n into the formula for $a_{\overline{n}|}$:

$$a_{\overline{n}|} = \frac{1 - v^n}{i} = \frac{1 - 0.5}{i} = 10$$

Thus, $i = \boxed{\mathbf{5\%}}$.

It is easy to show that $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$. First, a purely algebraic approach:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{\frac{1-v^{2n}}{i}}{\frac{1-v^n}{i}} = \frac{1 - v^{2n}}{1 - v^n}$$

Now, $1 - v^{2n}$ is the difference between two squares, which can be factored into $(1 + v^n)(1 - v^n)$.
Substituting in the above:

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{(1 + v^n)(1 - v^n)}{1 - v^n} = 1 + v^n$$

Another way to derive this result is by a general reasoning approach. Think of $a_{\overline{2n}|}$ as consisting of two annuities: (1) an immediate annuity for n years, followed by (2) a deferred annuity for another n years. The relationship is:

$$a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} = a_{\overline{n}|}(1 + v^n)$$

Hence $a_{\overline{2n}|}/a_{\overline{n}|} = 1 + v^n$.

It would be a good idea to know both of these approaches. It will help you to remember the above relationship and similar ones.

A couple of additional points:

1. In Section 4e, we will show that "double dots cancel." This means that the quotient $\ddot{a}_{\overline{2n}|}/\ddot{a}_{\overline{n}|}$ is the same as the quotient $a_{\overline{2n}|}/a_{\overline{n}|}$:

$$\frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}} = \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

   (More advanced Chapter 4 annuities with "upper m's" in their symbols also have the same quotient. See Section 4e.)

2. Formulas can also be derived for other annuities, for example, $a_{\overline{3n}|}/a_{\overline{n}|}$. Think of $a_{\overline{3n}|}$ as consisting of 3 annuities: an immediate annuity for n years, followed by a deferred annuity for n years, followed by another deferred annuity for n years. You should then be able to visualize the following relationship:

$$\begin{aligned} a_{\overline{3n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|} \\ &= a_{\overline{n}|}(1 + v^n + v^{2n}) \end{aligned}$$

   so that

$$\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}.$$

Example 3.25

Given that $a_{\overline{10}|} = X$ and $a_{\overline{20}|} = 1.6X$, determine i.

Solution

$$a_{\overline{20}|}/a_{\overline{10}|} = 1.6X/X = 1.6 = 1 + v^{10}$$
$$v^{10} = 0.6$$
$$i = \boxed{\mathbf{5.24\%}} \quad \text{(2 decimals)}$$

Example 3.26

The present value of a 20-year annuity-immediate with annual payments of 5 is equal to the present value of a 20-year annuity-immediate with annual payments of 3 for the first 10 years and 8 for the next 10 years. Determine i.

Solution

We are given that:

$$5a_{\overline{20}|} = 8a_{\overline{20}|} - 5a_{\overline{10}|}$$
$$a_{\overline{20}|}/a_{\overline{10}|} = 5/3 = 1.6\dot{6} = 1 + v^{10}$$
$$v^{10} = 0.6\dot{6}$$
$$i = \boxed{\mathbf{4.14\%}} \quad \text{(2 decimals)}$$

Example 3.27

Given that $a_{\overline{n}|} = 10$ and $a_{\overline{3n}|} = 20$, determine i.

Solution

$$a_{\overline{3n}|}/a_{\overline{n}|} = 20/10 = 2 = 1 + v^n + v^{2n}$$

This is a quadratic in v^n. For simplicity, let $v^n = x$:

$$x^2 + x - 1 = 0$$

The positive root is $x = .618034 = v^n$. Since $a_{\overline{n}|} = (1 - v^n)/i$, we have $10 = (1 - .618034)/i$ and $i = \boxed{\mathbf{3.82\%}}$ (2 decimals).

Summary of Concepts and Formulas in Section 3g

1. $$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

2. This can be derived algebraically or by using the fact that the PV of a $2n$-year annuity is the sum of the PV of an n-year annuity and the PV of an n-year deferred n-year annuity:

$$\begin{aligned} a_{\overline{2n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} \\ &= (1 + v^n) a_{\overline{n}|} \end{aligned}$$

3. Similar formulas can be derived for $3n$-year annuities, etc. For example:

$$\begin{aligned} a_{\overline{3n}|} &= a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|} \\ &= (1 + v^n + v^{2n}) a_{\overline{n}|} \end{aligned}$$

4. These formulas can be used to solve for i and n; for example, when numerical values are given for $a_{\overline{n}|}$ and $a_{\overline{2n}|}$.

Past Exam Questions on Section 3g

1. At an annual effective interest rate of i, $i > 0$, both of the following annuities have a present value of X:

   (i) a 20-year annuity-immediate with annual payments of 55

   (ii) a 30-year annuity-immediate with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years

   Calculate X. [5/03 #33]

   (A) 575 (B) 585 (C) 595 (D) 605 (E) 615

2. Dottie receives payments of X at the end of each year for n years. The present value of her annuity is 493.

   Sam receives payments of $3X$ at the end of each year for $2n$ years. The present value of his annuity is 2748.

   Both present values are calculated at the same annual effective interest rate.

   Determine v^n. [SOA 5/98 #4]

   (A) 0.86 (B) 0.87 (C) 0.88 (D) 0.89 (E) 0.90

3. An annuity pays 2 at the end of each year for 18 years.

   Another annuity pays 2.5 at the end of each year for 9 years.

   At an effective annual interest rate of i, $0 < i < 1$, the present values of both annuities are equal.

   Calculate i. [SOA 11/92 #5]

   (A) 14% (B) 17% (C) 20% (D) 23% (E) 26%

4. You are given:

   (i) $a_{\overline{n}|} = 10.00$; and

   (ii) $a_{\overline{3n}|} = 24.40$.

   Determine $a_{\overline{4n}|}$. [SOA 11/90 #2]

   (A) 28.74 (B) 29.00 (C) 29.26 (D) 29.52 (E) 29.78

5. Eric receives 12000 from a life insurance policy. He uses the fund to purchase two different annuities, each costing 6000.

   The first annuity is a 24-year annuity-immediate paying K per year to himself. The second annuity is an 8-year annuity-immediate paying $2K$ per year to his son.

   Both annuities are based on an annual effective interest rate of i, $i > 0$.

   Determine i. [SOA 11/90 #7]

   (A) 6.0% (B) 6.2% (C) 6.4% (D) 6.6% (E) 6.8%

6. At an effective annual interest rate i, you are given:

   (i) the present value of an annuity-immediate with annual payments of 1 for n years is 40; and

   (ii) the present value of an annuity-immediate with annual payments of 1 for $3n$ years is 70.

   Calculate the accumulated value of an annuity-immediate with annual payments of 1 for $2n$ years. [SOA 5/90 #6]

   (A) 240 (B) 243 (C) 260 (D) 268 (E) 280

7. Samantha receives a 10,000 life insurance benefit. If she uses the proceeds to buy an n-year immediate annuity, the annual payout will be 1,538. If a $2n$-year immediate annuity is purchased, the annual payout will be 1,072. Both calculations are based on an effective annual interest rate of i.

   Calculate i. [SOA 5/88 #16]

   (A) 0.087 (B) 0.088 (C) 0.089 (D) 0.090 (E) 0.091

8. At a rate of interest, i, where $i > 0$, a 36-year annuity-immediate with annual payments of \$4 has the same present value as an 18-year annuity-immediate with annual payments of \$5.

   In how many years does money double at rate of interest, i? [SOA SAMPLE/84 #8]

   (A) 9 (B) 12 (C) 18 (D) 27 (E) 36

9. An annuity-due has the following present value and accumulated value:

$$\ddot{a}_{\overline{n+2}|} = 13.987$$
$$\ddot{s}_{\overline{n}|} = 51.632$$

   Which of the following is closest to the effective annual rate of interest? [CAS 5/83 #7]

   (A) 5.2% (B) 5.4% (C) 5.6% (D) 5.8% (E) 6.0%

Solutions to Past Exam Questions on Section 3g

1.
$$X = 55a_{\overline{20}|} = 30a_{\overline{10}|} + 60v^{10}a_{\overline{10}|} + 90v^{20}a_{\overline{10}|}$$
$$= a_{\overline{10}|}\left(30 + 60v^{10} + 90v^{20}\right)$$

Dividing by $a_{\overline{10}|}$, we have:

$$55\frac{a_{\overline{20}|}}{a_{\overline{10}|}} = 55\left(1 + v^{10}\right) = 30 + 60v^{10} + 90v^{20}$$
$$90v^{20} + 5v^{10} - 25 = 0$$

This is a quadratic in v^{10}:

$$v^{10} = \frac{-5 + \sqrt{25 - (4)(90)(-25)}}{180} \text{ (positive root)}$$
$$= \frac{-5 + \sqrt{9025}}{180} = \frac{90}{180} = 0.5$$
$$\therefore i = 7.177346\%$$

$$X = 55a_{\overline{20}|i} = \boxed{\mathbf{574.97}} \qquad \text{ANS. (A)}$$

2.
$$\text{Dottie: } Xa_{\overline{n}|} = 493$$
$$\text{Sam: } 3Xa_{\overline{2n}|} = 2{,}748$$
$$\text{Dividing: } \frac{3Xa_{\overline{2n}|}}{Xa_{\overline{n}|}} = 3\left(1 + v^n\right) = \frac{2{,}748}{493}$$
$$v^n = \frac{2{,}748}{(3)(493)} - 1 = \boxed{\mathbf{.858}} \qquad \text{ANS. (A)}$$

3.
$$2a_{\overline{18}|} = 2.5a_{\overline{9}|}$$
$$\frac{2a_{\overline{18}|}}{a_{\overline{9}|}} = 2\left(1 + v^9\right) = 2.5$$
$$v^9 = \frac{2.5}{2} - 1 = .25,\ i = \boxed{\mathbf{16.65\%}} \qquad \text{ANS. (B)}$$

4. Dividing (ii) by (i):

$$\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n} = \frac{24.40}{10} = 2.44$$
$$v^{2n} + v^n - 1.44 = 0,\ v^n = \frac{-1 + \sqrt{1 - (4)(-1.44)}}{2} = 0.8$$
$$a_{\overline{4n}|} = a_{\overline{n}|} + v^n a_{\overline{3n}|} = 10 + (0.8)(24.4) = \boxed{\mathbf{29.52}} \qquad \text{ANS. (D)}$$

5.
$$Ka_{\overline{24}|} = 2Ka_{\overline{8}|};\ \text{Dividing:}$$
$$\frac{a_{\overline{24}|}}{a_{\overline{8}|}} = 2 = 1 + v^8 + v^{16}$$
$$v^{16} + v^8 - 1 = 0,\ v^8 = \frac{-1 + \sqrt{1 - (4)(-1)}}{2}$$
$$= \frac{\sqrt{5} - 1}{2} = .618034$$
$$i = \boxed{\mathbf{6.2\%}} \qquad \text{ANS. (B)}$$

6.
$$a_{\overline{n}|} = 40,\ a_{\overline{3n}|} = 70$$
$$\frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = \frac{7}{4} = 1 + v^n + v^{2n}$$
$$v^{2n} + v^n - \frac{3}{4} = 0,\ v^n = \frac{-1 + \sqrt{1 - (4)\left(-\frac{3}{4}\right)}}{2} = 0.5$$
$$a_{\overline{n}|} = 40 = \frac{1 - v^n}{i} = \frac{1 - 0.5}{i},\ i = \frac{0.5}{40} = .0125$$
$$s_{\overline{2n}|} = \frac{(1+i)^{2n} - 1}{i} = \frac{4 - 1}{.0125} = \boxed{\mathbf{240}} \qquad \text{ANS. (A)}$$

7.
$$1{,}538a_{\overline{n}|} = 1{,}072a_{\overline{2n}|}$$
$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n = \frac{1{,}538}{1{,}072} = 1.434701$$

and $v^n = .434701$. Substituting v^n:

$$10{,}000 = 1{,}538a_{\overline{n}|} = \frac{1{,}538(1 - .434701)}{i},$$
$$i = \frac{(1{,}538)(.565299)}{10{,}000} = \boxed{\mathbf{.0870}} \qquad \text{ANS. (A)}$$

8.
$$4a_{\overline{36}|} = 5a_{\overline{18}|}$$
$$\frac{a_{\overline{36}|}}{a_{\overline{18}|}} = 1 + v^{18} = \frac{5}{4}, \ v^{18} = \frac{1}{4},$$
$$(1+i)^{18} = 4, \ \boxed{\mathbf{(1+i)^9 = 2}} \qquad \text{ANS. (A)}$$

9.
$$\ddot{a}_{\overline{n+2}|} = 1 + a_{\overline{n+1}|} = 13.987, \ a_{\overline{n+1}|} = 12.987$$
$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1 = 51.632, \ s_{\overline{n+1}|} = 52.632$$
$$\frac{s_{\overline{n+1}|}}{a_{\overline{n+1}|}} = \frac{52.632}{12.987} = 4.05267 = (1+i)^{n+1}$$
$$s_{\overline{n+1}|} = \frac{(1+i)^{n+1} - 1}{i} = \frac{4.05267 - 1}{i} = 52.632$$
$$i = \frac{3.05267}{52.632} = \boxed{\mathbf{.058}} \qquad \text{ANS. (D)}$$

Part II

Six Original Practice Exams

Practice Exam 1

Note to Students: These practice exams follow the format of the actual exams in October 2022 and subsequent: 30 questions in $2\frac{1}{2}$ hours. The actual exam will be in CBT format. A few of the questions will be pilot questions that will not be graded, but you will have no way of knowing which ones they are.

When you take these exams, stick to the time limit and simulate exam conditions.

Questions for Practice Exam 1

1. Which of the following is *not* correct with respect to an annual effective interest rate of $i = 10\%$?

   (A) $\delta = e^{0.10} - 1$

   (B) $i^{(2)} = 2 \times [(1.10)^{0.50} - 1]$

   (C) $\delta = \ln(1.10)$

   (D) $d = \frac{0.10}{1.10}$

   (E) $d^{(4)} = 4 \times [1 - (1.10^{-0.25})]$

2. You can receive one of the following two sets of cash flows. Under Option A, you will receive 10 annual payments of \$1,000, with the first payment to occur 4 years from now. Under Option B, you will receive X at the end of each year, forever, with the first payment to occur 1 year from now. The annual effective rate of interest is 8%. Which of the following equations should be solved to find the value of X such that you are indifferent between these two options?

   (A) $80a_{\overline{10}|}v^4 = X$ (B) $80a_{\overline{13}|}v^3 = X$ (C) $80a_{\overline{10}|}v^3 = X$ (D) $80a_{\overline{10}|}v^3(0.08) = X$ (E) $80a_{\overline{13}|}v^2 = X$

3. An annuity will pay you \$500 two years from now, and another \$1,000 four years from now. The present value of these two payments is \$1,200. Find the implied effective annual interest rate, i.

   (A) $i \le 4.5\%$ (B) $4.5\% < i \le 5.5\%$ (C) $5.5\% < i \le 6.5\%$ (D) $6.5\% < i \le 7.5\%$ (E) $7.5\% < i$

4. An investor took out a 30-year loan which he repays with annual payments of 1,500 at an annual effective interest rate of 4%. The payments are made at the end of the year. At the time of the 12^{th} payment, the investor pays an additional payment of 4,000 and wants to repay the remaining balance over 10 years. Calculate the revised annual payment.

   (A) 1,682 (B) 1,729 (C) 1,783 (D) 1,825 (E) 1,848

5. A 25-year loan is being paid off via level amortization payments made at the end of each quarter. The nominal annual interest rate is 12% convertible monthly. The amount of principal in the 29th payment is 1,860. Find the amount of principal in the 61st payment

   (A) 4,535 (B) 4,635 (C) 4,735 (D) 4,835 (E) 4,935

6. Suppose you are the actuary for an insurance company. Your company, in response to a policyholder claim involving physical injury, is responsible for making annual medical payments. The first payment will occur on January 1, 2008, and the final payment will occur on January 1, 2031. The first payment will be \$100,000; after that, the payments will increase annually for inflation, at a rate of 5% per year. The real interest rate is 3% per year. Find the present value of these future payments as of December 31, 2005.

   (A) 1,491,000 (B) 1,501,000 (C) 1,511,000 (D) 1,521,000 (E) 1,531,000

7. A company must pay the following liabilities at the end of the years shown:

End of Year	Liability
2	\$1,000
4	X
6	1,000

The company achieves Redington immunization by purchasing assets that have two cash inflows: \$733 at the end of one year and Y at the end of 5 years. The effective annual rate of interest is 10%. Determine Y.

(A) 1,789 (B) 1,934 (C) 2,152 (D) 2,201 (E) 2,376

8. An investment is expected to pay 2 one year from now, and 3 two years from now. Thereafter, payments are annual with each being g% greater than the previous payment. The effective annual interest rate is 8.5%, and the purchase price of this investment is 112.50. Find g.

(A) 5.6 (B) 5.7 (C) 5.8 (D) 5.9 (E) 6.0

9. At any moment t, a continuously-varying continuous 5-year annuity makes payments at the rate of t^2 per year at moment t. The force of interest is 6%. Which of the following represents a correct expression of the present value of this annuity?

(A) $\int_0^5 t^2 e^{0.06t} dt$

(B) $\int_0^5 t^2 e^{-0.06t} dt$

(C) $\int_0^5 te^{-0.12t} dt$

(D) $\int_0^5 t^2 (1.06)^{-t} dt$

(E) None of (A), (B), (C), or (D) is a correct expression of the present value of the annuity.

10. A loan of 45,000 is being repaid with level annual payments of 3,200 for as long as necessary plus a final drop payment. All payments are made at the end of the year. The principal portion of the 9$^{\text{th}}$ payment is 1.5 times the principal portion of the 2$^{\text{nd}}$ payment. Calculate the drop payment.

(A) 1,495 (B) 1,521 (C) 1,546 (D) 1,584 (E) 1,597

11. A project requires an investment of 50,000 now (time 0), and will provide returns of X at the end of each of years 3 through 10. The effective annual rate of interest is 10%. The net present value of this project is 2,500. Find X.

(A) 11,300 (B) 11,500 (C) 11,700 (D) 11,900 (E) 12,100

12. Two growing perpetuities have the same yield rate. The first perpetuity—a perpetuity-immediate—has an initial payment of 500 one year from now, and each subsequent annual payment increases by 4%. This first perpetuity has a present value of 9,500. The second perpetuity—also a perpetuity-immediate—has an initial payment of 400 one year from now, and each subsequent annual payment increases by 20. Find the present value, P, of this second perpetuity.

(A) $P \le 6{,}500$

(B) $6{,}500 < P \le 6{,}600$

(C) $6{,}600 < P \le 6{,}700$

(D) $6{,}700 < P \le 6{,}800$

(E) $6{,}800 < P$

13. Jenna purchased an n-year \$1,000 par value bond at a discount to yield 4.2% convertible semiannually. The bond pays coupons at 3.6% convertible semiannually and has a redemption value of \$1,150. The purchase price is \$1,035. Calculate n.

(A) 6 (B) 8 (C) 12 (D) 16 (E) 24

14. A 10-year 200,000 loan is being paid off with level amortization payments at the end of each month. The effective annual interest rate is 15%. Find the amount of interest in the 56th monthly payment.

(A) 1,576 (B) 1,607 (C) 1,652 (D) 1,714 (E) 1,789

15. A 30-year \$300,000 loan involves level amortization payments at the end of each year. The effective annual interest rate is 9%. Let P be the ratio of total dollars of interest paid by the borrower divided by total aggregate payment dollars made by the borrower over the life of the loan. Find P.

(A) $P \leq 0.525$ (B) $0.525 < P \leq 0.575$ (C) $0.575 < P \leq 0.625$ (D) $0.625 < P \leq 0.675$ (E) $0.675 < P$

16. At the end of each year, for the next 19 years, you make deposits into an account, as follows:

$$\text{Deposit at end of year } t = 100t \quad \text{for } t = 1, 2, 3, \ldots, 10.$$
$$\text{Deposit at end of year } t = 1{,}000 - \{100(t-10)\} \quad \text{for } t = 11, 12, 13, \ldots, 19$$

The effective annual interest rate is 10%. Find the present value, at time $t = 0$, of this annuity.

(A) 4,053 (B) 4,103 (C) 4,153 (D) 4,203 (E) 4,253

17. An investment opportunity has the following characteristics: payments of \$10,000 will be made to you and invested into a fund at the beginning of each year, for the next 20 years. These payments will earn a 7% effective annual rate, and the interest payments (paid at the end of each year) will immediately be reinvested into a second account earning a 4% effective annual rate. Find the purchase price of this investment opportunity, given that it has an annual yield of 6% over the 20-year life of the investment.

(A) 92,000 (B) 102,000 (C) 112,000 (D) 122,000 (E) 132,000

18. A 30-year bond with par value 1,000 has annual coupons and sells for 1,300. The write down in the first year is 4.60. What is the yield-to-maturity for this bond?

(A) 4.73% (B) 4.89% (C) 4.98% (D) 5.15% (E) 5.27%

19. A \$7,600 loan is being repaid by level installments at the end of each year for 14 years. The annual effective rate of interest is 4% for the first 6 years and 5% thereafter. Which of the following formulas gives the amount of the level installment?

(A) $\dfrac{7,600}{a_{\overline{6}|4\%} + a_{\overline{8}|5\%}}$ (B) $\dfrac{7,600}{a_{\overline{14}|5\%} - a_{\overline{6}|4\%}}$ (C) $\dfrac{7,600}{a_{\overline{14}|4\%} - a_{\overline{8}|5\%}}$ (D) $\dfrac{7,600}{a_{\overline{6}|4\%}(1.05)^8 + a_{\overline{8}|5\%}}$ (E) $\dfrac{7,600}{a_{\overline{6}|4\%} + a_{\overline{8}|5\%}(1.04)^{-6}}$

20. A 20-year 100 par value bond with 8% semiannual coupons is purchased for 108.50. What is the book value of the bond just after the 13^{th} coupon is paid?

(A) 102.24 (B) 103.32 (C) 104.89 (D) 105.73 (E) 106.91

21. Yield rates to maturity for zero coupon bonds are currently quoted at 6% for one-year maturity, 7% for two-year maturity, and 7.5% for three-year maturity. Find the present value, two years from now, of a one-year 1000-par-value zero-coupon bond.

(A) 902 (B) 922 (C) 942 (D) 962 (E) 982

22. Determine the modified duration (or "volatility") of a growing perpetuity. The perpetuity will make annual payments, with the first payment being \$1 one year from now, and thereafter each subsequent payment will be \$1 greater than the preceding payment. Assume an annual effective interest rate of 8%.

(A) 12 (B) 16 (C) 20 (D) 24 (E) 28

23. You purchase a 7.5% annual coupon bond with a face value of 1,000 to yield a minimum interest rate of 8% effective. The bond is a callable corporate bond, with a call price of 1,050, and can be called by the issuing corporation after five years. The bond matures at par in 30 years. Immediately after the 12th coupon payment, the issuing corporation redeems the bond. Determine the effective annual yield you achieved on this twelve-year investment.

(A) 6.5% (B) 7.0% (C) 7.5% (D) 8.0% (E) 8.5%

24. A one-year zero-coupon bond has an annual yield of 6.25%. A two-year zero-coupon bond has an annual yield of 7.00%. A three-year zero-coupon bond has an annual yield of 7.50%. A three-year 12% annual coupon bond has a face value of \$1,000. Find the yield to maturity on this three-year 12% annual coupon bond.

(A) 6.6% (B) 7.0% (C) 7.4% (D) 7.8% (E) 8.2%

25. Bond A is an n-year 100 par value bond with 8% annual coupons and sells for 140.25. Bond B is an n-year 100 par value bond with 3% annual coupons and sells for 80.17. Both bonds have the same yield rate i. Determine i.

(A) 3.82% (B) 4.65% (C) 4.85% (D) 5.15% (E) 5.52%

26. A 30-year 1,000 par value bond pays 10% annual coupons. Using an interest rate of 12%, find the Macaulay duration of this bond.

(A) 9.2 (B) 10.2 (C) 11.2 (D) 12.2 (E) 13.2

27. An insurer must pay 3,000 and 4,000 at the ends of years 1 and 2, respectively. The only investments available to the company are a one-year zero-coupon bond (with a par value of 1,000 and an effective annual yield of 5%), and a two-year 8% annual coupon bond (with a par value of 1,000 and an effective annual yield of 6%). Which of the following is closest to the cost to the company today to match its liabilities exactly?

(A) 6,014 (B) 6,114 (C) 6,214 (D) 6,314 (E) 6,414

28. Sue decided to purchase a 20-year annuity that pays $900 at the end of the first year, $915 at the end of the second year, and for each year thereafter the payment increases by $15. Which of the following formulas gives the price of this annuity?

(A) $900 + 15(Ia)_{\overline{19}|}$ (B) $885 + 15(Ia)_{\overline{20}|}$ (C) $900a_{\overline{20}|} + 15(Ia)_{\overline{20}|}$ (D) $900a_{\overline{20}|} + 15(Ia)_{\overline{19}|}$ (E) $885a_{\overline{20}|} + 15(Ia)_{\overline{20}|}$

29. Christine deposits $100 into an account which earns interest at an effective annual rate of discount of d. At the same time, Douglas deposits $100 into a separate account earning interest at a force of interest of $\delta_t = 0.001t^2$. After 10 years, both accounts have the same value. Find d.

(A) 3.3% (B) 3.6% (C) 3.9% (D) 4.2% (E) 4.5%

30. You are given the following information about two annual-coupon bonds, each with a face and redemption value of $ 1,000, and each 3 years in length:

- Bond A: A 3-year 6% annual coupon bond with a price of $955.57.
- Bond B: A 3-year 8% annual coupon bond with a price of $1,008.38.

Using this data, find the annual yield on a 3-year zero-coupon bond.

(A) 6.40% (B) 6.95% (C) 7.30% (D) 7.85% (E) 8.40%

Solutions to Practice Exam 1

Question #	Answer
1	A
2	C
3	D
4	E
5	D
6	A
7	E
8	E
9	B
10	D
11	D
12	C
13	C
14	C
15	D

Question #	Answer
16	C
17	D
18	B
19	E
20	E
21	B
22	D
23	E
24	C
25	B
26	A
27	E
28	E
29	A
30	D

1. All of the formulas except the first (answer (A)) are valid equivalencies when the effective rate of interest is 10%. The correct relationship between the effective rate and the force of interest is $e^\delta = 1 + i$ or $i = e^\delta - 1$ or $\delta = \ln(1 + i)$. ANS. (A)

2. "Indifference" between two alternatives means that a person considers the present values of the two options to be equal. Setting up this equivalency relationship:

$$\boldsymbol{1{,}000 \cdot a_{\overline{10}|.08} \cdot v^3 = \frac{X}{0.08}}$$

which is equivalent to answer (C). The three-year present value factor on the left-hand side is necessary because the first payment is four years away, and the annuity-immediate formula provides a PV one year prior to the first payment (leaving three more years of discounting to invoke). ANS. (C)

3. Set up the present value formula. The key is to recognize this as a quadratic in v^2:

$$\begin{aligned} 1{,}200 &= 500v^2 + 1{,}000v^4 \\ 10(v^2)^2 + 5v^2 - 12 &= 0 \\ v^2 &= \frac{-5 \pm \sqrt{25 + 480}}{20} = 0.873610 \\ v &= 0.934671 \\ i &= \boxed{\mathbf{0.0699}} \qquad \text{ANS. (D)} \end{aligned}$$

4. The outstanding balance at time 12 prior to the additional payment is:

$$B_{12} = 1{,}500a_{\overline{18}|.04} = 18{,}988.95.$$

After the additional payment, the outstanding balance is 14,988.95.

To pay this remaining balance in 10 years, the revised annual payment is such that:

$$Pa_{\overline{10}|.04} = 14{,}988.95 \quad \text{that gives } P = \boxed{\mathbf{1{,}848.00}} \qquad \text{ANS. (E)}$$

5. The key in this problem is to use the $(1 + i)$ multiplicative factor relationship between the principal components of sequential amortization payments. This is a consequence of the formula $P_t = R \cdot v^{n-t+1}$. Thus, once the appropriate interest rate is determined, the answer can be found quickly:

$$\begin{aligned} j &= (1.01)^3 - 1 = 0.030301/qtr \\ P_{61} &= P_{29} \cdot (1 + j)^{32} = \boxed{\mathbf{4{,}834.65}} \qquad \text{ANS. (D)} \end{aligned}$$

6. This is an application of a geometrically-growing annuity present value function. It can be done using either real payments and interest rates, or nominal payments and rates. Using the latter approach:

$$\begin{aligned} i_{\text{NOM}} &= (1.05 \times 1.03) - 1 = 0.0815 \\ PV &= v_i \cdot 100{,}000 \cdot \left(\frac{1 - \left(\frac{1.05}{1.0815}\right)^{24}}{0.0815 - 0.05} \right) = \boxed{\mathbf{1{,}491{,}363}} \qquad \text{ANS. (A)} \end{aligned}$$

7. The first condition of Redington immunization is $P_A = P_L$, where P_A is the PV of the assets and P_L is the PV of the liabilities:

$$(1) \quad 733v + Yv^5 = 1000v^2 + Xv^4 + 1000v^6$$

Dividing (1) by v:

$$(2) \quad 733 + Yv^4 = 1000v + Xv^3 + 1000v^5$$

The second condition is $P'_A = P'_L$:

$$(3) \quad -733v^2 - 5Yv^6 = -2000v^3 - 4Xv^5 - 6000v^7$$

Dividing (3) by $-v^2$:

$$(4) \quad 733 + 5Yv^4 = 2000v + 4Xv^3 + 6000v^5$$

Multiplying (2) by 4:

$$(5) \quad 2932 + 4Yv^4 = 4000v + 4Xv^3 + 4000v^5$$

